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THE EXISTENCE OF A 3-CYCLE IMPLIES THE EXISTENCE OF ALL CYCLES (ANOTHER PROOF)

Tariq A. Al-Fadhel
alfadhel@ksu.edu.sa
Department of Mathematics, College of Science,
King Saud University, Saudi Arabia.

Abstract. It is known that the existence of a 3-cycle of a continuous map implies the existence of all cycles as in [Devaney,1992], [Sarkovskii,1964] and [Yorke and Li,1975]. This paper presents another technique using the inverse images of the fixed point of the map to give another proof of this fact and to give details about the phase portraits of each cycle.

Keywords. fixed point; 3-cycle; n-cycle; inverse image; phase portraits.

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1 Introduction

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function and $a_1 < a_2 < a_3$ be a three distinct real numbers such that $a_1 \to a_2 \to a_3 \to a_1$ a 3-cycle of f. In 1964 Alexandre Sarkovskii published a paper that includes an ordering of natural numbers and a theorem that bears his name, a special case of that theorem states that the existence of a 3-cycle implies the existence of all other cycles. See [4]. In 1975 James Yorke and Tien-Yien Li gave another proof of that fact. Moreover, they proved that the existence of a 3-cycle implies the existence of a dense orbit which leads to chaos. See [5]. In [2] Devaney gave another proof of this fact using the following two observations:

> **Observation (1):** Suppose $I = [a, b]$ and $J = [c, d]$ are closed intervals and $I \subset J$. If $J \subset f(I)$, then f has a fixed point in I.
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> **Observation (2):** Suppose I and J are two closed intervals and $J \subset f(I)$. Then there is a closed subinterval $I' \subset I$ such that $f(I') = J$.

Section two of this paper shows the existence of a fixed point of the map f, finds the inverse images of it and shows how these inverse images partitions the interval $[a_1, a_3]$. The inverse images of the fixed point play an important role of the existence of the other cycles and their phase portraits. Sections three and four use the inverse images of the fixed point of f and the two observations used by Devaney to show the existence of all other cycles and their phase portraits. Section five represents a demonstration of these facts by a