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DECAY RESULTS FOR NONLINEAR VISCOELASTIC PROBLEM WITH VELOCITY DEPENDENT MATERIAL IN THE PRESENT OF INFINITE MEMORY

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Abstract. In this paper we extend some general decay results, known for the case of finite history for a quasilinaer viscoelastic problem, to the case of infinite history where the relaxation function satisfies $g'(t) \leq -\xi(t)g^p(t), \quad t \geq 0, \quad 1 \leq p < 2$. Moreover, we delete some assumptions on the boundedness of initial data used in many earlier papers in the literature.

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AMS (MOS) subject classification: 35B40. 74D99. 93D15. 93D20.

1 Introduction

Problems related to viscoelasticity have attracted a great deal of attention and several papers of existence and long-time behavior have been published. Cavalcanti [5] discussed the following problem

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds - \gamma\Delta u_t = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \Gamma \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary Γ, ρ is a positive real number such that $0 < \rho \leq \frac{2}{n-2}$ if $n \geq 3$ and $\rho > 0$ if $n = 1, 2$, and g is a positive exponentially decaying function. They proved a global existence result for $\gamma \geq 0$ and an exponential decay result for the case $\gamma > 0$. Messaoudi and Tatar [28, 27] considered (1), for $\gamma = 0$, and showed that the solution energy decays exponentially (resp. polynomially) if g decays exponentially (resp. polynomially). Later, Han and Wang [14] considered (1) for $\gamma = 0$ and with a relaxation function of more general decay type, and established, similarly to the work of Messaoudi [20], a general decay result, from which the usual exponential and polynomial decay are only special cases. Messaoudi and Mustafa [26] considered (1) for relaxation functions satisfying a relation of the form

$$g'(t) \leq -H(g(t)), \tag{2}$$

where H is a convex function satisfying some smoothness conditions. They established a general relation between the decay rate for the energy and that of the relaxation function g without imposing restrictive assumptions on the behavior of g at infinity. Messaoudi and Al-Gharabli [22] studied (1) in the presence of infinite memory and with a relaxation function satisfying $g'(t) \leq -\xi(t)g(t)$. They established a general decay result depending on the relaxation function g. In [7], Cavalcanti et al. considered (1) with a relaxation function satisfying (2) and the additional requirement:

$$\lim \inf_{x\to 0^+} x^2 H''(x) - xH'(x) + H(x) \geq 0,$$

and that $y^{1-\alpha_0} \in L^1(1,\infty)$, for some $\alpha_0 \in [0,1)$, where $y(t)$ is the solution of the problem

$$y'(t) + H(y(t)) = 0, \; y(0) = g(0) > 0.$$

They characterized the decay of the energy by the solution of a corresponding ODE as in [16]. Recently, Messaoudi and Al-Khulaifi [25] treated (1) with a relaxation function satisfying (11) (below). They obtained a more general stability result for which the results of [21, 20] are only special cases. Moreover, the optimal decay rate for the polynomial case is achieved without any extra work and conditions as in [15] and [16]. For more results related to problem (1), we refer the reader to Liu [17, 18].
For infinite history problems, Giorgi et al. [9] considered the following semilinear hyperbolic equation, in a bounded domain $\Omega \subset \mathbb{R}^3$,

$$u_{tt} - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)ds + g(u) = f$$

with $K(0), K(\infty) > 0$ and $K' \leq 0$ and gave the existence of global attractors for the solutions. Conti and Pata [8] considered the following semilinear hyperbolic equation with linear memory in a bounded domain $\Omega \subset \mathbb{R}^n$,

$$u_{tt} + \alpha u_t - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)ds + g(u) = f \quad \text{in } \Omega \times \mathbb{R}^+ \tag{3}$$

where the memory kernel is a convex decreasing smooth function such that $K(0) > K(\infty) > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a nonlinear function of at most cubic growth satisfying some conditions. They proved the existence of a regular global attractor. In [3], Appleby et al. studied the linear integro-differential equation

$$u_{tt} + Au(t) + \int_{-\infty}^{t} K(t-s)Au(s)ds = 0, \quad t > 0$$

and established results of exponential decay of strong solutions in a Hilbert space. Pata [30] discussed the decay properties of the semigroup generated by the following equation

$$u_{tt} + \alpha Au(t) + \beta u_t(t) - \int_0^{+\infty} \mu(s)Au(t-s)ds = 0$$

where A is a strictly positive self-adjoint linear operator and $\alpha > 0$, $\beta \geq 0$ and the memory kernel μ is a decreasing function satisfying specific conditions. He established the necessary as well as the sufficient conditions for the exponential stability. In [10], Guesmia considered

$$u_{tt} + Au - \int_0^{+\infty} g(s)Bu(t-s)ds = 0$$

and introduced a new ingenuous approach for proving a more general decay result based on the properties of convex functions and the use of the generalized Young inequality. He used a larger class of infinite history kernels satisfies the following condition

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))}ds + \sup_{s\in\mathbb{R}^+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty, \tag{4}$$

such that

$$G(0) = G'(0) = 0 \text{ and } \lim_{t\to+\infty} G'(t) = +\infty, \tag{5}$$

where $G : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing strictly convex function. Using this approach, Guesmia and Messaoudi [13] later looked into

$$u_{tt} - \Delta u + \int_0^t g_1(t-s)div(a_1(x)\nabla u(s))ds + \int_0^{+\infty} g_2(s)div(a_2(x)\nabla u(t-s))ds = 0$$

in a bounded domain and under suitable conditions on a_1, a_2 and for a wide class of relaxation functions g_1 and g_2 which are not necessarily decaying polynomially or exponentially. They established a general decay result from which the usual exponential and polynomial decay rates are only special cases. Al-Mahdi and Al-Gharabli [2] considered the following viscoelastic problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta u(t-s)ds + |u_t|^{m-2}u_t = 0, & \text{in } \Omega \times (0,+\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,-t) = u_0(x,t),\ u_t(x,0) = u_1(x), & \text{in } \Omega \times (0,+\infty), \end{cases} \tag{6}$$

and they established decay results in which the relaxation function h satisfies

$$g'(t) \leq -\xi(t)g^p(t), \quad \forall t \geq 0,\ 1 \leq p < \frac{3}{2}, \tag{7}$$

and they obtained a better decay rate than the one of [10] and [12]. Mustafa [29] conside the following coupled quasilinear system

$$\begin{cases} |u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \displaystyle\int_0^t g_1(s)\Delta u(t-s)ds + f_1(u,v) = 0 \\ |v_t|^{\rho} v_{tt} - \Delta v - \Delta v_{tt} + \int_0^{\infty} g_2(s)\Delta v(t-s)ds + f_2(u,v) = 0 \end{cases} \tag{8}$$

and established more general decay rate results where the relaxation functions satisfy $g_i'(t) \leq -H(g_i(t)),\ i = 1,2$. He provided more general decay rates for which the usual exponential and polynomial rates are only special cases. For more results with infinite history, we refer the reader to [22, 24, 23, 1, 4, 19, 6]. In this paper, our aim is to investigate the following problem

$$\begin{cases} |u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^{+\infty} g(s)\Delta u(t-s)ds = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \Gamma \times \mathbb{R}^+, \\ u(x,-t) = u_0, \quad u_t(x,0) = u_1(x), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{9}$$

for a relaxation function satisfying (11) (below) and obtain a general stability result for a wide class of kernels, among which those of the exponential decay type, are only special cases. Equation (9) is a nonlinear wave equation with the presence of a viscoelastic damping with infinite memory supplemented by a history function u_0 and initial data u_1. Here, Ω is a bounded domain of $\mathbb{R}^N (N \geq 1)$ with a smooth boundary Γ, u is the transverse displacement of waves, the relaxation function g is positive and decreasing, the exponent ρ is a positive real number satisfying some conditions to be specified later.
In the present work, we study the asymptotic behavior of solutions of (9), under the assumption (11)(below) instead of (4) considered in Guesmia [10] and Messaoudi and Al-Gharabli [22] and instead of (7) considered in Al-Mahdi and Al-Gharabli [2]. Moreover, our technique is different than the one in [29] and we deleted some assumptions on the boundedness of initial data used in many earlier papers in the literature. In fact, our results generalize, extend and improve many earlier results in the literature.
This paper is organized as follows. In section 2, we present some notations and material needed for our work. Some technical lemmas and the decay results are presented in section 3 and section 4, respectively.

2 Assumptions

In this section, we present some material needed for the proof of our result. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. For the relaxation function g, we assume the following

$(A1)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 decreasing function satisfying

$$g(0) > 0, \quad 1 - \int_0^{+\infty} g(s)ds = \ell > 0. \tag{10}$$

$(A2)$ There exists a nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$g'(t) \leq -\xi(t) g^p(t), \quad \forall\ 1 \leq p < 2,\ t \in \mathbb{R}^+. \tag{11}$$

$(A3)$ For the nonlinearity, we assume that

$$0 < \rho \leq \frac{2}{N-2},\ N \geq 3\ \textit{ and }\ \rho > 0,\ N = 1, 2. \tag{12}$$

The energy associated with problem (9) is

$$E(t) = \frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2} dx + \frac{\ell}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx + \frac{1}{2}(go\nabla u)(t), \tag{13}$$

where

$$(go\nabla u)(t) = \int_0^{+\infty} g(s) \|\nabla u(t-s) - \nabla u(t)\|_2^2 ds.$$

Direct differentiation, using (9), gives

$$E'(t) = \frac{1}{2}(g' o \nabla u)(t) \leq 0. \tag{14}$$

For completeness we state, without proof, the existence result which can be established exactly repeating the proof of [5].

Proposition 1 *Let $(u_0(.,0), u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be given. Assume that $(A1)-(A4)$ are satisfied; then problem (9) has a unique global (weak) solution*

$$u \in C^1(\mathbb{R}^+; H_0^1(\Omega)).$$

3 Technical Lemmas

In this section, we state and establish several lemmas needed for the proof of our main result.

Lemma 2 *There exists a positive constant M_1 such that*

$$\int_t^{+\infty} g(s) ||\nabla u(t) - \nabla u(t-s)||_2^2 ds \leq M_1 h_0(t), \tag{15}$$

where $h_0(t) = \int_0^{+\infty} g(t+s)\left(1 + ||\nabla u_0(s)||^2\right) ds$.

Proof. The proof is identical to the one in [11]. Indeed, we have

$$
\begin{aligned}
&\int_t^{+\infty} g(s)||\nabla u(t)-\nabla u(t-s)||_2^2 ds \\
&\le 2||\nabla u(t)||^2 \int_t^{+\infty} g(s)ds + 2\int_t^{+\infty} g(s)||\nabla u(t-s)||^2 ds \\
&\le 2\sup_{s\ge 0}||\nabla u(s)||^2 \int_0^{+\infty} g(t+s)ds + 2\int_0^{+\infty} g(t+s)||\nabla u(-s)||^2 ds \\
&\le \frac{4E(s)}{\ell}\int_0^{+\infty} g(t+s)ds + 2\int_0^{+\infty} g(t+s)||\nabla u_0(s)||^2 ds \\
&\le \frac{4E(0)}{\ell}\int_0^{+\infty} g(t+s)ds + 2\int_0^{+\infty} g(t+s)||\nabla u_0(s)||^2 ds \\
&\le M_1\int_0^{+\infty} g(t+s)\left(1+||\nabla u_0(s)||^2\right) ds.
\end{aligned}
\tag{16}
$$

where $M_1 = \max\left\{2, \frac{4E(0)}{\ell}\right\}$.

4 Decay of solution

In this section we state and prove the main result of our work.

Lemma 3 *[22] Assume that $(A1)-(A3)$ hold. Then there exist constants $\varepsilon, \alpha_1, \alpha_2, M > 0$ such that the functional*

$$L = ME + \varepsilon\chi_1 + \chi_2$$

satisfies, for all $t \in \mathbb{R}^+$,

$$L \sim E. \tag{17}$$

Moreover,

$$L'(t) \le -\alpha_1 E(t) + \alpha_2 (go\nabla u)(t), \tag{18}$$

where

$$\chi_1(t) := \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx + \int_\Omega \nabla u.\nabla u_t dx$$

and

$$\chi_2(t) := \int_\Omega \left(\Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1}\right)\int_0^{+\infty} g(s)(u(t)-u(t-s))dsdx$$

Theorem 4 *Let $(u_0(\cdot,0), u_1) \in H_0^1(\Omega)\times H_0^1(\Omega)$ be given. Assume that $(A1)-(A3)$ hold. Then there exist strictly positive constants C, δ_0, δ_1 such that the solution of (9) satisfies, for all $t > t_0$,*

$$E(t) \le \delta_1\left(1+\int_0^t (g(s))^{1-\delta_0} ds\right) e^{-\delta_0\int_0^t \xi(s)ds} + \delta_1\int_t^{+\infty} g(s)ds, \qquad p=1, \tag{19}$$

and

$$E(t) \leq C(1+t)^{\frac{-1}{p-1}} \xi^{-\frac{p}{p-1}}(t) \left(1 + \int_0^t (1+s)^{\frac{1}{p-1}} \xi^{\frac{p}{p-1}}(s) h(s) ds\right), \qquad 1 < p < 2, \tag{20}$$

where $h(t) = \xi^{\alpha+1}(t) \left(h_0\right)^{\alpha+1}(t)$ *and* $h_0(t)$ *is defined in Lemma 3.1.*

Proof. The proof of (19) is similar to the one in [22]. For the proof of (20), we introduce the following

$$\begin{aligned} I(t) &= \frac{q}{t-t_0} \int_0^t \|\nabla u(t) - \nabla u(s)\|^2 ds \leq \frac{cq}{t-t_0} \int_0^t E(s) ds \\ &\leq \frac{cq}{t-t_0} \int_0^t E(0) ds < +\infty. \end{aligned} \tag{21}$$

where q is small enough. We see that, for some positive constant c, (15) and (18) lead to

$$\begin{aligned} L'(t) \leq & \quad -cE(t) + cI(t) \cdot \frac{1}{I(t)} \left(g^{p \cdot \frac{1}{p}} \circ \nabla u\right)(t) + cM_1 h_0(t) \\ \leq & \quad -cE(t) + cI(t) \left(\frac{1}{I(t)} g^p \circ \nabla u\right)^{\frac{1}{p}}(t) + cM_1 h_0(t) \\ \leq & \quad -cE(t) + cI^{1-\frac{1}{p}}(t) \left(\frac{-g'}{\xi} \circ \nabla u\right)^{\frac{1}{p}}(t) + cM_1 h_0(t) \\ \leq & \quad -cE(t) + \frac{c}{[\xi(t)]^{\frac{1}{p}}} \left(-g' \circ \nabla u\right)^{\frac{1}{p}}(t) + cM_1 h_0(t) \\ \leq & \quad -cE(t) + \frac{c}{[\xi(t)]^{\frac{1}{p}}} \left[-E'(t)\right]^{\frac{1}{p}} + cM_1 h_0(t). \end{aligned} \tag{22}$$

Multiply both sides of (22) by $\xi^{\alpha+1} E^{\alpha}$ where $p = \alpha + 1$, we get

$$\begin{aligned} \xi^{\alpha+1}(t) E^{\alpha}(t) L'(t) \leq & -c\xi^{\alpha+1}(t) E^{\alpha+1}(t) + c\left(\xi^{\frac{\alpha}{\alpha+1}}(t)\right)\left(\xi^{\alpha} E^{\alpha}\right)(t)(-E')^{\frac{1}{\alpha+1}}(t) \\ & + cM_1 h_0(t) \xi^{\alpha+1}(t) E^{\alpha}(t). \end{aligned} \tag{23}$$

Since $\frac{\alpha}{\alpha+1} > 0$ and $\xi(t) \leq \xi(0)$, (23) becomes

$$\begin{aligned} \xi^{\alpha+1} E^{\alpha}(t) L'(t) \leq & -c\xi^{\alpha+1}(t) E^{\alpha+1}(t) \\ & + c\left(\xi^{\alpha} E^{\alpha}\right)(t)(-E')^{\frac{1}{\alpha+1}}(t) + cM_1 h_0(t) \xi^{\alpha+1}(t) E^{\alpha}(t). \end{aligned} \tag{24}$$

Using Young's inequality with $\gamma = \alpha + 1$ and $\gamma' = \frac{\alpha+1}{\alpha}$ in the second and third terms of (24), we get

$$\begin{aligned} \xi^{\alpha+1}(t) E^{\alpha}(t) L'(t) \leq & -c\xi^{\alpha+1}(t) E^{\alpha+1}(t) + \varepsilon\left(\xi^{\alpha+1} E^{\alpha+1}\right)(t) - c_{\varepsilon} E'(t) \\ & + c\xi^{\alpha+1}(t) \varepsilon E^{\alpha+1}(t) + c_{\varepsilon}\left(h_0(t)\right)^{\alpha+1}. \end{aligned} \tag{25}$$

Therefore, we have

$$\xi^{\alpha+1}(t)E^{\alpha}(t)L'(t)+c_{\varepsilon}E'(t) \leq -(c-\varepsilon-c\varepsilon)\xi^{\alpha+1}(t)E^{\alpha+1}(t)+c_{\varepsilon}\xi^{\alpha+1}(t)\left(h_0(t)\right)^{\alpha+1}. \tag{26}$$

Choosing ε small enough, letting $F := \xi^{\alpha+1}E^{\alpha}L + c_{\varepsilon}E \sim E$ and $h(t) = \xi^{\alpha+1}(t)\left(h_0(t)\right)^{\alpha+1}$, we have for a positive constants c_1 and c_2,

$$F'(t) \leq -c_1\xi^{\alpha+1}F^{\alpha+1} + c_2 h(t). \tag{27}$$

Multiply both sides of (27) by ξ^{β}, for $\beta > 1$, then we have

$$\xi^{\beta}(t)F'(t) \leq -c_1\xi^{\alpha+1+\beta}(t)F^{\alpha+1}(t) + c_2\xi^{\beta}(t)h(t). \tag{28}$$

Recalling that $\xi > 0$ and $\xi' \leq 0$, one can have

$$(\xi^{\beta}F)'(t) \leq -c_1\xi^{\alpha+1+\beta}(t)F^{\alpha+1}(t) + c_2\xi^{\beta}h(t), \tag{29}$$

noting $\varphi = \xi^{\beta}F$ and taking $\beta = \frac{\alpha+1}{\alpha}$, we obtain

$$\varphi'(t) \leq -c_1\varphi^{\alpha+1}(t) + c_2\xi^{\beta}(t)h(t). \tag{30}$$

Let

$$f(t) := \varphi(t)-\Psi(t); \text{ where } \Psi(t) = c_2(1+t)^{\frac{-1}{\alpha}}\int_0^t \xi^{\beta}(s)h(s)(1+s)^{\frac{1}{\alpha}}ds. \tag{31}$$

From the definition of Ψ, we have

$$c_2\xi^{\beta}(t)h(t) = \Psi'(t) + \frac{c_2}{\alpha}(1+t)^{\frac{-1}{\alpha}-1}\int_0^t \xi^{\beta}(s)h(s)(1+s)^{\frac{1}{\alpha}}ds \tag{32}$$

and for all $t \geq t_0 > 0$,

$$\nu := \int_0^{t_0} \xi^{\beta}(s)h(s)(1+s)^{\frac{1}{\alpha}}ds \leq \int_0^t \xi^{\beta}(s)h(s)(1+s)^{\frac{1}{\alpha}}ds$$

and then

$$\frac{\int_0^t \xi^{\beta}(s)h(s)(1+s)^{\frac{1}{\alpha}}ds}{\nu} \geq 1, \quad \forall t \geq t_0.$$

Thus, (32) yields, $\forall t \geq t_0$,

$$c_2\xi^{\beta}(t)h(t) \leq \Psi'(t)+\frac{1}{\alpha c_2^{\alpha}\nu^{\alpha}}c_2^{\alpha+1}\left[(1+t)^{\frac{-1}{\alpha}}\right]^{\alpha+1}\left[\int_0^t \xi^{\beta}(s)h(s)(1+s)^{\frac{1}{\alpha}}ds\right]^{\alpha+1}. \tag{33}$$

We can choose c_2 large enough so that $\frac{1}{\alpha c_2^{\alpha}\nu^{\alpha}} \leq c_1$, and then we get

$$c_2\xi^{\beta}(t)h(t) \leq \Psi'(t) + c_1\Psi^{\alpha+1}, \quad \forall t \geq t_0. \tag{34}$$

Now using (34), (30) and the definition of f in (31), we get, $\forall t \geq t_0$,

$$\begin{aligned} f'(t) = \varphi'(t) - \Psi'(t) &\leq -c_1 \varphi^{\alpha+1}(t) + c_2 \xi^{\beta}(t) h(t) - \Psi'(t) \\ &\leq -c_1 \left[(f+\Psi)^{\alpha+1}(t) \right] + c_2 \xi^{\beta}(t) h(t) - \Psi'(t). \end{aligned} \tag{35}$$

Since $f(0) > 0$. Then there exists $t_1 > 0$ such that $f(t) > 0, \forall t \in [0, t_1)$. Hence,

$$\begin{aligned} f'(t) &\leq -c_1 \left[f^{\alpha+1}(t) + \Psi^{\alpha+1}(t) \right] + c_2 \xi^{\beta}(t) h(t) - \Psi'(t) \\ &\leq -c_1 \left[f^{\alpha+1}(t) + \Psi^{\alpha+1}(t) - \frac{c_2}{c_1} \xi^{\beta}(t) h(t) + \frac{1}{c_1} \Psi'(t) \right], \ \forall\, t \in [t_0, t_1). \end{aligned} \tag{36}$$

Thus,

$$f'(t) \leq -c_1 f^{\alpha+1}(t), \quad \forall\, t \in [t_0, t_1). \tag{37}$$

Integrate over (t_0, t), we have

$$f(t) \leq \frac{c}{(t-t_0)^{\frac{1}{\alpha}}}, \quad \forall\, t \in [t_0, t_1). \tag{38}$$

If $t_1 = +\infty$, using again the definitions of f and Ψ, we have, for t large enough,

$$\varphi(t) \leq C(1+t)^{\frac{-1}{\alpha}} \left[1 + \int_0^t \xi^{\beta}(s) h(s) (1+s)^{\frac{1}{\alpha}} ds \right]. \tag{39}$$

If $t_1 < +\infty$, then there exists $t_2 > t_1$ such that $f(t) \leq 0, \forall\, t_1 \leq t < t_2$. Hence, (31) yields $\varphi(t) \leq \Psi(t), \forall\, t_1 \leq t < t_2$, consequently, we get (39). If $t_2 = +\infty$, we are done. Otherwise, there exists $t_3 > t_2$ such that $f(t_2) = 0$ and $f(t) > 0, \forall\, t_2 < t < t_3$, we then repeat the steps (36)-(38) on $[t_2, t_3)$ to obtain (39). Therefore, (39) remains valid for all $t \geq t_0$. Multiply (39) by $\xi^{-\beta}$ and recall the definition of φ, then for $\beta = \frac{\alpha+1}{\alpha}$ we have, for t large enough

$$F(t) \leq C(1+t)^{\frac{-1}{\alpha}} \xi^{-\frac{\alpha+1}{\alpha}}(t) \left[1 + \int_0^t \xi^{\frac{\alpha+1}{\alpha}}(s) h(s) (1+s)^{\frac{1}{\alpha}} ds \right] \tag{40}$$

Using the fact $F \sim E$, and recalling that $\alpha = p - 1$, we get

$$E(t) \leq C(1+t)^{\frac{-1}{p-1}} \xi^{-\frac{p}{p-1}}(t) \left(1 + \int_0^t (1+s)^{\frac{1}{p-1}} \xi^{\frac{p}{p-1}}(s) h(s) ds \right). \tag{41}$$

This establishes (20).

The following example illustrates our results:

Example: Let $g(t) = \frac{a}{(1+t)^q}$, $q > 2$, where a is chosen so that hypothesis $(A1)$ holds, then

$$g'(t) = \frac{-aq}{(1+t)^{q+1}} = -b \left(\frac{a}{(1+t)^q} \right)^{\frac{q+1}{q}} = -b g^p(t), \quad p = \frac{q+1}{q} < 2, \quad b > 0 \tag{42}$$

We will discuss two cases:
Case 1: if $m_0 \leq 1 + ||\nabla u_0||^2 \leq m_1$. Then we have the following:

$$h(t) = \left(\xi(t) \int_0^{+\infty} g(t+s)(1 + ||\nabla u_0(s)||^2) ds\right)^p \leq c(1+t)^{p(1-q)}, \ \ q = \frac{1}{p-1}. \tag{43}$$

Routine calculations yield, for some positive constant C,

$$\int_0^t (1+s)^{\frac{1}{2p-1}} \xi^{\frac{2p}{2p-1}}(s) h(s) ds \leq C(1+t)^{p(1-q)+\frac{1}{2p-1}+1}. \tag{44}$$

Therefore, the estimate (20) yields

$$E(t) \leq C(1+t)^{\frac{-(q-1)(q-2)(q+1)}{q(q+2)}}. \tag{45}$$

Case 2: if $m_0(1+t)^r \leq 1 + ||\nabla u_0||^2 \leq m_1(1+t)^r$, where $0 < r < q-1$, then we have

$$c_1(1+t)^{-(q-r-1)} \leq \int_0^{+\infty} g(t+s)(1+||\nabla u_0(s)||^2) ds \leq c_2(1+t)^{-(q-r-1)}. \tag{46}$$

Then

$$h(t) = \left(\xi(t) \int_0^{+\infty} g(t+s)(1 + ||\nabla u_0(s)||^2) ds\right)^p \leq c_2(1+t)^{-p(q-r-1)} \tag{47}$$

Therefore, the estimate (20), yields for any $0 < r < q-1$,

$$\begin{aligned} E(t) \leq & C(1+t)^{\frac{-1}{2p-1}} \left(1 + (1+t)^{-p(q-r-1)}\right) \\ &= C(1+t)^{-q} + C(1+t)^{-(pq-pr-p+q)} \\ &= C(1+t)^{-q} + C(1+t)^{-(2q+1-pr-p)} \\ &\leq (1+t)^{\frac{-q}{q+2}}. \end{aligned} \tag{48}$$

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References

[1] Al-Gharabli, M. M. A general decay result of a viscoelastic equation with infinite history and nonlinear damping. *Applicable Analysis 97*, 3 (2018), 382–399.

[2] Al-Mahdi, A. M., and Al-Gharabli, M. M. New general decay results in an infinite memory viscoelastic problem with nonlinear damping. *Boundary Value Problems 2019*, 1 (2019), 140.

[3] Appleby, J. A., Fabrizio, M., Lazzari, B., and Reynolds, D. W. On exponential asymptotic stability in linear viscoelasticity. *Mathematical Models and Methods in Applied Sciences 16*, 10 (2006), 1677–1694.

[4] Boukhatem, Y., and Benabderrahmane, B. Asymptotic behavior for a past history viscoelastic problem with acoustic boundary conditions. *Applicable Analysis* (2018), 1–21.

[5] Cavalcanti, M., Domingos Cavalcanti, V., and Ferreira, J. Existence and uniform decay for a non-linear viscoelastic equation with strong damping. *Mathematical methods in the applied sciences 24*, 14 (2001), 1043–1053.

[6] Cavalcanti, M., Guesmia, A., et al. General decay rates of solutions to a nonlinear wave equation with boundary condition of memory type. *Differential and Integral equations 18*, 5 (2005), 583–600.

[7] Cavalcanti, M. M., Cavalcanti, V. N. D., Lasiecka, I., and Webler, C. M. Intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density. *Advances in Nonlinear Analysis 6*, 2 (2017), 121–145.

[8] Conti, M., and Pata, V. Weakly dissipative semilinear equations of viscoelasticity. *Communications on Pure and Applied Analysis 4*, 4 (2005), 705.

[9] Giorgi, C., Rivera, J. E. M., and Pata, V. Global attractors for a semilinear hyperbolic equation in viscoelasticity. *Journal of Mathematical Analysis and Applications 260*, 1 (2001), 83–99.

[10] Guesmia, A. Asymptotic stability of abstract dissipative systems with infinite memory. *Journal of Mathematical Analysis and Applications 382*, 2 (2011), 748–760.

[11] Guesmia, A. New general decay rates of solutions for two viscoelastic wave equations with infinite memory. *(in press), Mathematical Modelling and Analysis journal* (2019).

[12] Guesmia, A., and Messaoudi, S. A new approach to the stability of an abstract system in the presence of infinite history. *Journal of Mathematical Analysis and Applications 416*, 1 (2014), 212–228.

[13] Guesmia, A., and Messaoudi, S. A. A general decay result for a viscoelastic equation in the presence of past and finite history memories. *Nonlinear Analysis: Real World Applications 13*, 1 (2012), 476–485.

[14] Han, X., and Wang, M. General decay of energy for a viscoelastic equation with nonlinear damping. *Mathematical Methods in the Applied Sciences 32*, 3 (2009), 346–358.

[15] Lasiecka, I., Messaoudi, S. A., and Mustafa, M. I. Note on intrinsic decay rates for abstract wave equations with memory. *Journal of Mathematical Physics 54*, 3 (2013), 031504.

[16] Lasiecka, I., Tataru, D., et al. Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. *Differential Integral Equations 6*, 3 (1993), 507–533.

[17] Liu, W. General decay rate estimate for a viscoelastic equation with weakly nonlinear time-dependent dissipation and source terms. *Journal of Mathematical Physics 50*, 11 (2009), 113506.

[18] Liu, W. General decay and blow-up of solution for a quasilinear viscoelastic problem with nonlinear source. *Nonlinear Analysis: Theory, Methods & Applications 73*, 6 (2010), 1890–1904.

[19] Liu, W., Wang, D., and Chen, D. General decay of solution for a transmission problem in infinite memory-type thermoelasticity with second sound. *Journal of Thermal Stresses 41*, 6 (2018), 758–775.

[20] Messaoudi, S. A. General decay of solutions of a viscoelastic equation. *Journal of Mathematical Analysis and Applications 341*, 2 (2008), 1457–1467.

[21] Messaoudi, S. A. General decay of the solution energy in a viscoelastic equation with a nonlinear source. *Nonlinear Analysis: Theory, Methods & Applications 69*, 8 (2008), 2589–2598.

[22] Messaoudi, S. A., and Al-Gharabli, M. M. A general stability result for a nonlinear wave equation with infinite memory. *Applied Mathematics Letters 26*, 11 (2013), 1082–1086.

[23] Messaoudi, S. A., and Al-Gharabli, M. M. A general decay result of a nonlinear system of wave equations with infinite memories. *Applied Mathematics and Computation 259* (2015), 540–551.

[24] Messaoudi, S. A., and Al-Gharabli, M. M. A general decay result of a viscoelastic equation with past history and boundary feedback. *Zeitschrift für angewandte Mathematik und Physik 66*, 4 (2015), 1519–1528.

[25] Messaoudi, S. A., and Al-Khulaifi, W. General and optimal decay for a quasilinear viscoelastic equation. *Applied Mathematics Letters 66* (2017), 16–22.

[26] Messaoudi, S. A., and Mustafa, M. I. A general stability result for a quasilinear wave equation with memory. *Nonlinear Analysis: Real World Applications 14*, 4 (2013), 1854–1864.

[27] Messaoudi, S. A., and Tatar, N.-e. Global existence and uniform stability of solutions for a quasilinear viscoelastic problem. *Mathematical methods in the applied sciences 30*, 6 (2007), 665–680.

[28] Messaoudi, S. A., and Tatar, N.-e. Exponential and polynomial decay for a quasilinear viscoelastic equation. *Nonlinear Analysis: Theory, Methods & Applications 68*, 4 (2008), 785–793.

[29] Mustafa, M. I. Energy decay in a quasilinear system with finite and infinite memories. In *Mathematical Methods in Engineering*. Springer, 2019, pp. 235–256.

[30] Pata, V. Stability and exponential stability in linear viscoelasticity. *Milan journal of mathematics 77*, 1 (2009), 333.

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