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BEHAVIOUR OF ORBITS AND PERIODS OF A TWO DIMENSIONAL DYNAMICAL SYSTEM ASSOCIATED TO A SPECIAL QRT-MAP

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Abstract. We study some general properties of the special QRT-dynamical system in $(\mathbb{R}_*)^2$,

$$\begin{cases} Xx = \dfrac{ey^2+gy+h}{y^2+by+d} \\ Yy = \dfrac{dX^2+fX+h}{X^2+cX+e}, \end{cases} \tag{1}$$

associated to the family of T-invariant biquadratic curves $\mathcal{C}(K)$ with equation

$$x^2y^2+bx^2y+cxy^2+dx^2+ey^2+fx+gy+h-Kxy=0. \tag{2}$$

Then, we investigate a typical particular symmetric case where a phenomenon occurs in the behavior of orbits and periods of this dynamical system. This phenomenon depends on the existence of some equilibrium point of the system, which is a saddle point for the corresponding invariant function of the map T. As a typical example, we study the family of curves $C_d(K)$ with equations

$$x^2y^2-5xy(x+y)+d(x^2+y^2)-20(x+y)+16-Kxy=0. \tag{3}$$

We precisely describe the type of behavior in each class corresponding to some case of variation for K : convergence, divergence, periods of periodic orbits, density of these orbits, sensitiveness to initial conditions...

Keywords. Difference equations; elliptic curves; QRT-maps; rotation number; periodic orbits.

AMS (MOS) subject classification: 39A20,39A11 primary; `14H52,14H70` secondary.

1 Introduction

It is nowadays a well-known fact that QRT-maps play a relevant role in the realm of discrete (integrable) systems and birational geometry. It is well established belief that these maps as well their generalizations and reductions deserves their own interest. In the present paper, we study the dynamical system $(x,y)\mapsto T(x,y)=(X,Y)$ defined if $x\neq 0$ and $y\neq 0$ by

$$\begin{cases} Xx = \dfrac{ey^2+gy+h}{y^2+by+d} \\ Yy = \dfrac{dX^2+fX+h}{X^2+cX+e}. \end{cases}$$