*Dynamics of Continuous, Discrete and Impulsive Systems*
*Series* **A:** *Mathematical Analysis* **25 (2018)** 93-110


# ARBITRARY DECAY RESULT OF A VISCOELASTIC EQUATION WITH INFINITE MEMORY AND NONLINEAR FRICTIONAL DAMPING

Mohammad M. Al-Gharabli

King Fahd University of Petroleum and Minerals
The Preparatory Year Program, Department of Mathematics
Dhahran 31261, Saudi Arabia.
E-mail: mahfouz@kfupm.edu.sa

**Abstract.** In this paper, we consider a viscoelastic equation with a weak frictional damping in the presence of infinite-memory term and prove an explicit and general decay result using the multiplier method and some properties of the convex functions. Our result is obtained without imposing any restrictive growth assumption on the damping term and strongly weakening the usual assumptions on the relaxation function.

**Keywords.** General decay, Infinite memory, Viscoelastic, Convexity, Frictional damping

## 1. Introduction

In this paper, we consider the following viscoelastic problem:

$$
(1.1)\qquad \begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^{+\infty} g(s)\Delta u(x,t-s)ds + \sigma(t)h(u_t) = 0, \\ \qquad \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \quad \text{on } \partial\Omega \times (0,\infty) \\ u(x,-t) = u_0(x,t), u_t(x,0) = u_1(x), \ \ \text{in } \Omega \times (0,\infty) \end{cases}
$$

where $u$ denotes the transverse displacement of waves, $\Omega$ is a bounded domain of $\mathbb{R}^N (N \geq 1)$ with a smooth boundary $\partial\Omega$ and $g, h, \sigma$ are specific functions. During the last half century, viscoelastic problems were studied by several authors and many existence and long-time behavior results have been established. We start with the pioneer work of Dafermos [13], [14], where he considered the following one-dimensional viscoelastic problem

$$
\rho u_{tt} = c u_{xx} - \int_{-\infty}^{t} g(t-s) u_{xx} ds
$$

and established various existence results and then proved, for smooth monotone decreasing relaxation functions, that the solutions go to zero as $t$ goes to infinity. However, no rate of decay has been specified. Hrusa [21] considered a one-dimensional nonlinear viscoelastic equation of the form

$$u_{tt} - cu_{xx} + \int_0^t m(t-s)(\psi(u_x(x,s)))_x ds = f(x,t)$$

and proved several global existence results for large data. He also proved an exponential decay result for strong solutions when $m(s) = e^{-s}$ and $\psi$ satisfies certain conditions. In [15], Dassios and Zafiropoulos considered a viscoelatic problem in $\mathbb{R}^3$ and proved a polynomial deacy result for exponentially decaying kernels. In their book, Fabrizio and Morro [16] established a uniform stability of some problems in linear viscoelasticity. After that, Rivera [31] considered equations for linear isotropic homogeneous viscoelastic solids of integral type which occupy bounded domains or the whole space $\mathbb{R}^n$. In the bounded-domain case and for exponentially decaying memory kernels and regular solutions, he showed that the sum of the first and the second energy decays exponentially. For the whole-space case and for exponentially decaying memory kernels, he showed that the rate of decay of energy is of algebraic type and depends on the regularity of the solution. This result was later generalized to a situation, where the kernel is decaying algebraically but not exponentially by Cabanillas and Rivera [6]. In their paper, the authors considered the case of bounded domains as well as the case when the material is occupying the entire space and showed that the decay of solutions is also algebraic, at a rate which can be determined by the rate of the decay of the relaxation function. This latter result was later improved by Baretto et al. [3], where equations related to linear viscoelastic plates were treated. Precisely, they showed that the solution energy decays at the same decay rate of the relaxation function. For partially viscoelastic materials, Rivera and Salvatierra [38] showed that the energy decays exponentially, provided the relaxation function decays in a similar fashion and the dissipation is acting on a part of the domain near to the boundary. Also, Rivera et al. [35], [36] established the same result as in [38] regardless to the size of the viscoelastic part of the material. Fabrizio and Polidoro [17] studied the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t = 0, & \text{in } \Omega \times (0,\infty) \\ u = 0, & \text{on } \partial\Omega \times (0,\infty) \end{cases}$$

and showed that the exponential decay of the relaxation function is a necessary condition for the exponential decay of the solution energy. In [34], a class of abstract viscoelastic equations of the form

$$u_{tt} + Au(t) + \beta u(t) - (g * A^\alpha u)(t) = 0, \tag{1.2}$$

for $0 \le \alpha \le 1$ and $\beta \ge 0$, was investigated. The main focus was on the case when $0 < \alpha < 1$ and the main result was that solutions for (1.2) decay polynomially even if the kernel g decay exponentially. This result has been generalized by Rivera et al. [33], where the authors studied a more general abstract problem than (1.2) and established a necessary and sufficient condition to obtain an exponential decay.

For quasilinear problems, Cavalcanti et al. [7] studied, in a bounded domain, the following equation

$$(1.3) \qquad |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma\Delta u_t = 0,$$

for $\rho > 0$. A global existence result for $\gamma \geq 0$, as well as an exponential decay result for $\gamma > 0$, have been established. This latter result was then extended to a situation, where $\gamma = 0$, by Messaoudi and Tatar [29, 30], and exponential and polynomial decay results have been established in the absence, as well as in the presence, of a source term. In all the above mentioned works, the rates of decay in relaxation functions were either of exponential or polynomial type. Very recently, Messaoudi and Mustafa [28] considered (1.3), for relaxation functions satisfying

$$g'(t) \leq -H(g(t))$$

for some positive convex function $H$. They used the properties of the convex functions together with the generalized Young inequality and established a general decay result depending on $g$ and $H$. For more general decaying relaxation functions, Messaoudi [25, 26] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = b|u|^{p-2}u$$

for $p \geq 2$ and $b \in \{0, 1\}$, and established a more general decay result, from which the usual exponential and polynomial decay rates are only special cases. In [10], Cavalcanti et al. considered

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + a(x)u_t + |u|^{p-1}u = 0, \quad \text{in } \Omega \times (0,\infty)$$

where $a : \Omega \to \mathbb{R}^+$ is a function which may vanish on a part of the domain $\Omega$ but satisfies $a(x) \geq a_0$ on $\omega \subset \Omega$ and $g$ satisfies, for two positive constants $\xi_1$ and $\xi_2$,

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0$$

and established an exponential decay result under some restrictions on $\omega$. Berrimi and Messaoudi [4] established the result of [10], under weaker conditions on both $a$ and $g$, to a problem where a source term is competing with the damping term. Cavalcanti and Oquendo [11] considered the following problem

$$u_{tt} - k_0\Delta u + \int_0^t \operatorname{div}[a(x)g(t-s)\Delta u(x,s)]ds + b(x)h(u_t) + f(u) = 0$$

and established, for $a(x) + b(x) \geq \rho > 0$, an exponential stability result for $g$ decaying exponentially and $h$ linear, and a polynomial stability result for $g$ decaying polynomially and $h$ nonlinear.

For Frictional dissipative boundary condition, Lasiecka and Tataru [23] investigated problem (1.1) in the absence of the viscoelastic term (g=0) and, without imposing any growth condition on $h$, they proved that the energy decays as fast as the solution of an associated differential equation whose coefficients depend on the damping term. A nonlinear wave equation with viscoelastic boundary condition was also studied by Rivera and Andrade [32] and the existence and uniform decay results, under some restriction on the initial data, were established. Santos [39] considered a one-dimensional wave equation with viscoelastic boundary feedback and showed, under some assumptions on both $g'$ and $g''$, that the dissipation is strong enough to produce exponential (polynomial) decay of the solution, provided the relaxation function also decays exponentially (polynomially) respectively. Cavalcanti et al. [9] studied problem (1.1) but with finite memory and established a global existence of strong as well as weak solutions and some uniform decay results under quite restrictive assumptions on both the damping function $h$ and the kernel $g$. After that, Cavalcanti et al. [8] weakened the conditions on both $h$ and $g$ and established a uniform stability depending on the behavior of $h$ and $g$.

For infinite history problems, Giorgi et al. [18] considered the following semilinear hyperbolic equation, in a bounded domain $\Omega \subset \mathbb{R}^3$,

$$u_{tt} - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)ds + g(u) = f$$

with $K(0), K(\infty) > 0$ and $K' \leq 0$ and gave the existence of global attractors for the solutions. Conti and Pata [12] considered the following semilinear hyperbolic equation with linear memory in a bounded domain $\Omega \subset \mathbb{R}^n$ ,

$$u_{tt} + \alpha u_t - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)ds + g(u) = f \quad \text{in } \Omega \times \mathbb{R}^+ \tag{1.4}$$

where the memory kernel is a convex decreasing smooth function such that $K(0) > K(\infty) > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a nonlinear function of at most cubic growth satisfying some conditions and proved the existence of a regular global attractor. In [1], Appleby et al. studied the linear integro-differential equation

$$u_{tt} + Au(t) + \int_{-\infty}^{t} K(t-s)Au(s)ds = 0, \quad t > 0$$

and established results of exponential decay of strong solutions in a Hilbert space. Pata [37] discussed the decay properties of the semigroup generated by the following equation

$$u_{tt} + \alpha Au(t) + \beta u_t(t) - \int_0^{+\infty} \mu(s)Au(t-s)ds = 0$$

where $A$ is a strictly positive self-adjoint linear operator and $\alpha > 0$, $\beta \geq 0$ and the memory kernel $\mu$ is a decreasing function satisfying some specific conditions. He established the necessary as well as the sufficient conditions for the exponential stability. In [19], Guesmia considered

$$u_{tt} + Au - \int_0^{+\infty} g(s)Bu(t-s)ds = 0.$$

and, by introducing a new ingenuous approach based on the properties of convex functions which allows a larger class of infinite-history kernels than the one considered in the literature, he established a more general decay result for a class of hyperbolic problems. Using this approach, Guesmia and Messaoudi [20] later looked into

$$u_{tt}-\Delta u+\int_0^t g_1(t-s)div(a_1(x)\nabla u(s))ds+\int_0^{+\infty} g_2(s)div(a_2(x)\nabla u(t-s))ds=0$$

in a bounded domain and under suitable conditions on $a_1$, $a_2$ and for a wide class of relaxation functions $g_1$ and $g_2$ which are not necessarily decaying polynomially or exponentially, and established a general decay result from which the usual exponential and polynomial decay rates are only special cases. The rest of our paper is organized as follows. In section 2, we present some material needed to prove our result. Some technical lemmas and the statement with proof of the main result will be given in section 3.

## 2. Preliminaries

In this section, we present some materials needed in the proof of our results. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. Throughout this paper, c is used to denote a generic positive constant.

We consider the following hypotheses:

$(A1)$ $g:\mathbb{R}^+\to\mathbb{R}^+$ is a $C^1$ nonincreasing function satisfying

$$g(0)>0,\ 1-\int_0^{+\infty} g(s)ds=\ell>0 \tag{2.1}$$

and there exists a strictly increasing function $G:\mathbb{R}^+\to\mathbb{R}^+$ and strictly convex on $(0,r_1]$, for some $r_1>0$, of class $C^1(\mathbb{R}^+)\cap C^2(0,\infty)$ satisfying

$$G(0)=G'(0)=0 \text{ and } \lim_{t\to+\infty} G'(t)=+\infty \tag{2.2}$$

such that

$$\sup_{s\in\mathbb{R}^+}\frac{g(s)}{G^{-1}(-g'(s))}:=N_2<\infty \tag{2.3}$$

$$\int_0^{+\infty}\frac{g(s)}{G^{-1}(-g'(s))}ds:=N_3<\infty \tag{2.4}$$

$(A2)$ $h:\mathbb{R}\to\mathbb{R}$ is a nondecreasing $C^0$ function such that there exists a strictly increaing function $h_0\in C^1(\mathbb{R}^+)$, with $h_0(0)=0$, and positive constants $c_1, c_2, \varepsilon$ such that

$$\begin{aligned} &h_0(|s|)\le|h(s)|\le h_0^{-1}(|s|) && \text{for all } |s|\le\varepsilon\\ &c_1|s|\le|h(s)|\le c_2|s| && \text{for all } |s|\ge\varepsilon \end{aligned} \tag{2.5}$$

In addition, we assume that the function $H$, defined by $H(s) = \sqrt{s}h_0(\sqrt{s})$, is a strictly convex $C^2$ function on $(0, r_2]$, for some $r_2 > 0$, when $h_0$ is nonlinear.

$(A3)$ $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing differentiable function.

$(A4)$ There exists a positive constant $m_0$, such that

$$||\nabla u_0(s)||_2 \leq m_0, \ \forall s \geq 0. \tag{2.6}$$

**Remark 2.1.** *We can deduce using* $(A1)$ *that if* $g'(s_0) = 0$ *for some* $s_0 \geq 0$*, then* $g(s) = 0$ *for all* $s \geq s_0$*.*

**Remark 2.2.** *Hypothesis* $(A2)$ *implies that* $sh(s) > 0$*, for all* $s \neq 0$*.*

**Remark 2.3.** *By* $(A1)$*, we easily deduce that* $\lim_{t\to+\infty} g(t) = 0$*. This implies that*
$\lim_{t\to+\infty}(-g'(t))$ *can not be equal to a positive number, and so it is natural to assume that* $\lim_{t\to+\infty}(-g'(t)) = 0$*. Hence, there is* $t_1 > 0$ *large enough such that* $g(t_1) > 0$ *and*

$$\max\{g(t), -g'(t)\} < \min\{r_1, G(r_1)\}, \quad \forall t \geq t_1. \tag{2.7}$$

We introduce the "modified" energy associated to problem (1.1):

$$E(t) = \frac{1}{2}||u_t||_2^2 + \frac{1-\ell}{2}|| \ \nabla u||_2^2 + \frac{1}{2}(go\nabla u)(t) \tag{2.8}$$

where

$$(go\nabla u)(t) = \int_0^{+\infty} g(s)||\nabla u(t) - \nabla u(t-s)||_2^2 ds$$

Direct differentiation, using (1.1), leads to

$$E'(t) = \frac{1}{2}(g'o\nabla u)(t) - \sigma(t)\int_\Omega u_t h(u_t)dx \leq 0 \tag{2.9}$$

**Remark 2.4.** *By exploiting* (2.6)-(2.8)*, we obtain* $\forall t, s \in \mathbb{R}^+$

$$\begin{aligned} ||\nabla u(t) - \nabla u(t-s)||_2^2 &\leq 2||\nabla u(t)||_2{}^2 + 2||\nabla u(t-s)||_2^2 \\ &\leq 4\sup_{s>0}||\nabla u(s)||_2{}^2 + 2\sup_{\tau<0}||\nabla u(\tau)||_2{}^2 \\ &\leq 4\sup_{s>0}||\nabla u(s)||_2^2 + 2\sup_{\tau>0}||\nabla u_0(\tau)||_2{}^2 \\ &\leq \frac{8}{1-\ell}E(0) + 2m_0^2 := N_1 \end{aligned} \tag{2.10}$$

For completeness we state, without proof, the following standard existence and regularity result (see [24], [9]).

**Proposition 2.5.** *Let* $(u_0(.,0), u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume* $(A1) - (A4)$ *are satisfied, then problem* (1.1) *has a unique global (weak) solution*

$$u \in C(R^+, H_0^1(\Omega)) \cap C^1(R^+, L^2(\Omega)).$$

*Moreover, if*

$$(u_0(.,0),u_1) \in \left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times H_0^1(\Omega)$$

*then the solution satisfies*

$$u \in L^\infty\left(\mathbb{R}^+, H^2(\Omega)\cap H_0^1(\Omega)\right)\cap W^{1,\infty}\left(\mathbb{R}^+, H_0^1(\Omega)\right)\cap W^{2,\infty}\left(\mathbb{R}^+, L^2(\Omega)\right).$$

## 3. General decay

In this section we state and prove our main decay result which reads as follows:

**Theorem 3.1.** *Let $(u_0(.,0),u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ be given. Assume that $(A1)-(A4)$ hold. Then, there exist positive constants $k_2$ , $k_3$, $k_4$, $\delta_1$, $\varepsilon_0$ such that*

$$E(t)\le k_4 W_1^{-1}\left(k_2\int_0^t \sigma(s)ds+k_3\right),\quad \forall t\ge 0, \tag{3.1}$$

*where*

$$W_1(t)=\int_t^1 \frac{1}{W_2(s)}ds \quad \text{and} \quad W_2(t)=tG'\left(\delta_1 t\right)H'\left(\varepsilon_0 t\right)$$

The proof of Theorem 3.1 will be done through several Lemmas.

**Lemma 3.2.** *For $u\in H_0^1(\Omega)$, we have*

$$\int_\Omega\left(\int_0^{+\infty} g(s)(u(t)-u(t-s))ds\right)^2dx \le (1-\ell)C_p^2(go\nabla u)(t),$$

*where $C_p$ is the Poincaré constant.*

**Lemma 3.3.** *Under the assumptions $(A1)-(A3)$, the functional*

$$\psi(t):=\int_\Omega uu_t dx$$

*satisfies, along the solution, the estimate*

$$\psi'(t)\le -\frac{\ell}{2}||\ \nabla u||_2^2+||u_t||_2^2+c(go\nabla u)(t)+c\int_\Omega h^2(u_t)dx \tag{3.2}$$

*Proof.* Direct computations, using (1.1), yield

$$\begin{aligned}\psi'(t)&=\int_\Omega u_t^2dx+\int_\Omega u\Delta u dx-\int_\Omega u\int_0^{+\infty} g(s)\Delta u(t-s)dsdx\\&\quad-\sigma(t)\int_\Omega uh(u_t)dx\\&=\int_\Omega u_t^2dx-\ell\int_\Omega|\nabla u|^2dx-\sigma(t)\int_\Omega uh(u_t)dx\\&\quad+\int_\Omega \nabla u.\int_0^{+\infty} g(s)(\nabla u(t-s)-\nabla u(t))dsdx.\end{aligned} \tag{3.3}$$

Using Young's inequality and Lemma 3.2, we obtain

$$\begin{aligned}
&\int_\Omega \nabla u.\int_0^{+\infty} g(s)(\nabla u(t-s)-\nabla u(t))dsdx\\
&\le \delta\int_\Omega |\nabla u|^2dx+\frac{1}{4\delta}\int_\Omega\left(\int_0^{+\infty} g(s)|\nabla u(t-s)-\nabla u(t)|ds\right)^2dx\\
&\le \delta\int_\Omega |\nabla u|^2dx+\frac{c}{\delta}(go\nabla u)(t).
\end{aligned}\tag{3.4}$$

Also, the use of Young's and Poincaré's inequalities gives

$$\begin{aligned}
-\sigma(t)\int_\Omega uh(u_t)dx &\le c\delta\int_\Omega u^2dx+\frac{c}{4\delta}\int_\Omega h^2(u_t)dx\\
&\le c\delta\int_\Omega |\nabla u|^2dx+\frac{c}{4\delta}\int_\Omega h^2(u_t)dx.
\end{aligned}\tag{3.5}$$

Combining (3.3)-(3.5) and choosing $\delta$ small enough give (3.2). □

**Lemma 3.4.** *Under the assumptions* $(A1)-(A4)$*, the functional*

$$\chi(t):=-\int_\Omega u_t\int_0^{+\infty} g(s)(u(t)-u(t-s))dsdx \tag{3.6}$$

*satisfies, along the solution, the estimate*

$$\begin{aligned}
\chi'(t)\le \frac{\ell}{4}||\ \nabla u||_2^2-(1-\ell-\frac{\ell}{4})||u_t||_2^2+\frac{4c}{\ell}(go\nabla u)(t)\\
-\frac{4c}{\ell}(g'o\nabla u)(t)+c\int_\Omega h^2(u_t)dx
\end{aligned}\tag{3.7}$$

*Proof.* By differentiating (3.6), using (1.1), and performing integration by parts, we arrive at

$$\begin{aligned}
\chi'(t) &= \int_\Omega \nabla u . \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t)) ds dx \\
&- \int_\Omega \left( \int_0^{+\infty} g(s) \nabla u(t-s) ds \right) . \left( \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t)) ds \right) dx \\
&+ \int_\Omega \left( \int_0^{+\infty} g(s)(u(t-s) - u(t)) ds \right) h(u_t) dx \\
&- \int_\Omega u_t \int_0^{+\infty} g'(s)(u(t-s) - u(t)) ds dx - (1-\ell) \int_\Omega {u_t}^2 dx \\
&= \ell \int_\Omega \nabla u . \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t)) ds dx \\
&+ \int_\Omega \left| \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t)) ds \right|^2 dx \\
&+ \int_\Omega \left( \int_0^{+\infty} g(s)(u(t-s) - u(t)) ds \right) h(u_t) dx \\
&- \int_\Omega u_t \int_0^{+\infty} g'(s)(u(t-s) - u(t)) ds dx - (1-\ell) \int_\Omega u_t^2 dx.
\end{aligned}$$

Using Young's inequality and Lemma 3.2, we obtain

$$\ell \int_\Omega \nabla u . \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t)) ds dx \le \delta \int_\Omega |\nabla u|^2 dx + \frac{c}{\delta}(go\nabla u)(t)$$

$$\int_\Omega \left( \int_0^{+\infty} g(s)(u(t-s) - u(t)) ds \right) h(u_t) dx \le c(go\nabla u)(t) + c \int_\Omega h^2(u_t) dx$$

and

$$- \int_\Omega u_t \int_0^{+\infty} g'(s)(u(t-s) - u(t)) ds dx \le \delta \int_\Omega u_t^2 dx - \frac{c}{\delta}(g' o \nabla u)(t).$$

Combining all the above estimates and putting $\delta = \frac{\ell}{4}$, (3.7) is established. □

**Lemma 3.5.** *Assume that (A1) − (A4) hold. Then there exist constants* $M_1, M_2, m, c > 0$ *such that the functional*

$$L(t) = M_1 E(t) + M_2 \chi(t) + \psi(t)$$

*satisfies, for all* $t \in \mathbb{R}^+$,

$$L'(t) \le -mE(t) + c(go\nabla u)(t) + c \int_\Omega h^2(u_t) dx \tag{3.8}$$

*Proof.* By using (2.9), (3.2), (3.7), we easily see that

$$L'(t) \leq -\frac{\ell}{4}||\ \nabla u||_2^2 - \left(M_2\left(1-\ell-\frac{\ell}{4}\right)-1\right)||u_t||_2^2 + \left(\frac{4c}{\ell}M_2^2+c\right)(go\nabla u)(t)$$
$$+\left(\frac{1}{2}M_1 - \frac{4c}{\ell}M_2^2\right)(g'o\nabla u)(t) + (cM_2+c)\int_\Omega h^2(u_t)dx.$$

At this point, we choose $M_2$ large enough so that

$$\alpha := M_2\left(1-\ell-\frac{\ell}{4}\right)-1>0,$$

and then $M_1$ large enough that

$$\frac{1}{2}M_1 - \frac{4c}{\ell}M_2^2 > 0.$$

So, we arrive at

$$L'(t) \leq -\frac{\ell}{4}||\ \nabla u||_2^2 - \alpha||u_t||_2^2 + c(g'o\nabla u)(t) + c\int_\Omega h^2(u_t)dx \tag{3.9}$$

Therefore, (3.9) reduces to (3.8) for two positive constants $m$ and $c$. On the other hand (see [4]), we can choose $M_1$ even larger (if needed)
so that

$$L \sim E \tag{3.10}$$

□

**Lemma 3.6.** *Assume that (A1) and (A4) are satisfied. Then there exists* $\beta_1 > 0$ *such that for all* $\delta_0 < \frac{r_1}{E(0)}$ *and* $t \in \mathbb{R}^+$,

$$G'(\delta_0 E(t))(go\nabla u)(t) \leq -\beta_1 E'(t) + \beta_1\delta_0 E(t)G'(\delta_0 E(t)) \tag{3.11}$$

*Proof.* First, using Remark (2.1), we can assume without loss of generality that $g' < 0$.

Using (2.7) and (2.10), we easily prove that

$$t_1 := G^{-1}\left(\frac{-g'(s)}{N_1}||\nabla u(t) - \nabla u(t-s)||_2^2\right) \leq r_1$$

Now, define

$$t_2 := \frac{G'(\delta_0 E(t))g(s)||\nabla u(t) - \nabla u(t-s)||_2^2}{N_1 N_2 G^{-1}\left(-\tau_2 g'(s)||\nabla u(t) - \nabla u(t-s)||_2^2\right)}$$

Using the fact that $G^{-1}$ is concave and $G^{-1}(0) = 0$, the function $K(s) = \frac{s}{G^{-1}(s)}$ satisfies, for any $0 \leq s_1 < s_2$,

$$K(s_1) = \frac{s_1}{G^{-1}\left(\frac{s_1}{s_2}s_2 + \left(1-\frac{s_1}{s_2}\right)0\right)}$$
$$\leq \frac{s_1}{\frac{s_1}{s_2}G^{-1}(s_2) + \left(1-\frac{s_1}{s_2}\right)G^{-1}(0)} = \frac{s_2}{G^{-1}(s_2)} = K(s_2)$$

Therefore, using (2.10) and the fact that $K$ is nondecreasing, we get

$$
\begin{aligned}
&\frac{G'(\delta_0 E(t))g(s)||\nabla u(t)-\nabla u(t-s)||_2^2}{N_1N_2G^{-1}\left(\frac{-g'(s)}{N_1}||\nabla u(t)-\nabla u(t-s)||_2^2\right)}\\
&\qquad= \frac{G'(\delta_0 E(t))g(s)}{-N_2g'(s)}K\left(\frac{-g'(s)}{N_1}||\nabla u(t)-\nabla u(t-s)||_2^2\right)\\
&\qquad\le \frac{G'(\delta_0 E(t))g(s)}{-N_2g'(s)}K\left(-g'(s)\right)\\
&\qquad\le \frac{G'(\delta_0 E(t))g(s)}{N_2G^{-1}(-g'(s))}
\end{aligned} \tag{3.12}
$$

Using (2.3) and choosing $\delta_0 < \frac{r_1}{E(0)}$, inequality (3.12) gives

$$t_2 < G'(r_1) \tag{3.13}$$

Let $G^*$ be the convex conjugate of $G$ in the sense of Young (see [2] p.61-64); then

$$
\begin{aligned}
G^*(t) &= t(G')^{-1}(t) - G\left((G')^{-1}(t)\right)\\
&\le t(G')^{-1}(t), \qquad\qquad \forall t\in(0,G'(r_1)]
\end{aligned} \tag{3.14}
$$

Using the general Young inequality: $t_1t_2 \le G(t_1)+G^*(t_2)$, we get for all $t\in\mathbb{R}^+$,

$$
\begin{aligned}
(go\nabla u)(t) &= \int_0^{+\infty} g(s)||\nabla u(t)-\nabla u(t-s)||_2^2 ds\\
&= \frac{N_1N_2}{G'(\delta_0E(t))}\int_0^{\infty}\Bigg\{G^{-1}\left(\frac{-g'(s)}{N_1}||\nabla u(t)-\nabla u(t-s)||_2^2\right)\\
&\qquad\qquad \times\frac{G'(\delta_0 E(t))g(s)||\nabla u(t)-\nabla u(t-s)||_2^2}{N_1N_2G^{-1}\left(\frac{-g'(s)}{N_1}||\nabla u(t)-\nabla u(t-s)||_2^2\right)}\Bigg\}ds\\
&\le -\frac{N_2}{G'(\delta_0E(t))}(g'o\nabla u)(t)\\
&\quad+\frac{N_1N_2}{G'(\delta_0E(t))}\int_0^{\infty}G^*\left(\frac{G'(\delta_0 E(t))g(s)||\nabla u(t)-\nabla u(t-s)||_2^2}{N_1N_2G^{-1}\left(\frac{-g'(s)}{N_1}||\nabla u(t)-\nabla u(t-s)||_2^2\right)}\right)ds
\end{aligned}
$$

Combing (2.9) and (3.14), we have for all $t\in\mathbb{R}^+$,

$$(go\nabla u)(t) \le -\frac{2N_2}{G'(\delta_0 E(t))}E'(t) + \int_0^{+\infty}\left\{\frac{g(s)||\nabla u(t)-\nabla u(t-s)||_2^2}{G^{-1}\left(\frac{-g'(s)}{N_1}||\nabla u(t)-\nabla u(t-s)||_2^2\right)} \times (G')^{-1}\left(\frac{G'(\delta_0 E(t))g(s)||\nabla u(t)-\nabla u(t-s)||_2^2}{N_1N_2G^{-1}\left(\frac{-g'(s)}{N_1}||\nabla u(t)-\nabla u(t-s)||_2^2\right)}\right)\right\}ds \tag{3.15}$$

Therefore, using (2.10), (3.13) and the fact that $(G')^{-1}$ is nondecreasing on $(0, G'(r_1)]$, we get

$$(G')^{-1}\left(\frac{G'(\delta_0 E(t))g(s)||\nabla u(t)-\nabla u(t-s)||_2^2}{N_1N_2G^{-1}\left(\frac{-g'(s)}{N_1}||\nabla u(t)-\nabla u(t-s)||_2^2\right)}\right) \le (G')^{-1}\left(\frac{G'(\delta_0 E(t))g(s)}{N_2G^{-1}(-g'(s))}\right)$$

Thus, we obtain from (3.15) and (2.10) that, for all $t \in \mathbb{R}^+$,

$$(go\nabla u)(t) \le -\frac{2N_2}{G'(\delta_0 E(t))}E'(t) + N_1\int_0^{+\infty}\frac{g(s)}{G^{-1}(-g'(s))}(G')^{-1}\left(\frac{G'(\delta_0 E(t))g(s)}{G^{-1}(-g'(s))}\right)ds$$

Using (2.3) and (2.4) and recalling that $(G')^{-1}$ is nondecreasing on $(0, G'(r_1)]$, we obtain, for all $t \in \mathbb{R}^+$

$$\begin{aligned}(go\nabla u)(t) &\le -\frac{2N_2}{G'(\delta_0 E(t))}E'(t) + N_1(G')^{-1}\left(\tau_1 N_1 N_2 G'(\delta_0 E(t))\right)\int_0^{+\infty}\frac{g(s)}{G^{-1}(-g'(s))}ds \\ &\le -\frac{2N_2}{G'(\delta_0 E(t))}E'(t) + N_1N_3\delta_0 E(t),\end{aligned}$$

which gives (3.11) with $\beta_1 = max\{2N_2, N_1N_3\}$ □

**Proof of Theorem 3.1**

**Case 1**. $h_0$ is linear. Then, using $(A2)$ we have

$$c_1'|s| \le |h(s)| \le c_2'|s|, \quad \forall s \in \mathbb{R}$$

and hence

$$h^2(s) \le c_2' s h(s), \quad \forall s \in \mathbb{R} \tag{3.16}$$

Therefore, after multiplying (3.8) by $\sigma(t)$ and using (2.9) and (3.16), we obtain

$$\begin{aligned}\sigma(t)L'(t) &\le -m\sigma(t)E(t) + c\sigma(t)(go\nabla u)(t) + c\sigma(t)\int_\Omega u_t h(u_t)dx \\ &\le -m\sigma(t)E(t) + c\sigma(t)(go\nabla u)(t) - cE'(t), \qquad \forall t \ge 0\end{aligned}$$

Consequently, $F_0(t) := \sigma(t)L(t) + cE(t)$ satisfies

$$F_0'(t) \le -m\sigma(t)E(t) + c\sigma(t)(go\nabla u)(t) \tag{3.17}$$

and recalling (3.10) and $(A3)$, we have $F_0 \sim E$. Now, we multiply (3.17) by $G'(\delta_0 E(t))$ and use (3.11) to obtain:

$$\begin{aligned}G'(\delta_0 E(t))F_0'(t) &\le -m\sigma(t)G'(\delta_0 E(t))E(t) - c\beta_1\sigma(t)E'(t) \\ &\quad + c\beta_1\delta_0\sigma(t)E(t)G'(\delta_0 E(t)) \\ &= -(m - c\beta_1\delta_0)\sigma(t)E(t)G'(\delta_0 E(t)) - c\beta_1\sigma(t)E'(t)\end{aligned}$$

Choosing $\delta_0$ small enough so that $\beta_2 := m - c\beta_1\delta_0 > 0$ and put

$$F_1(t) := G'(\delta_0 E(t))F_0(t) + c\beta_1\sigma(t)E(t)$$

we deduce (note that $G'(\delta_0 E(t))$ is nonincerasing for $\delta_0 < \frac{r_1}{E(0)}$) that

$$F_1 \sim E \quad \text{ and } F_1'(t) \le -k_1\sigma(t)F_1(t)G'(\delta_1 F_1(t))$$

The last inequality implies that $(W_1(F_1))' \ge k_1\sigma(t)$, where

$$W_1(t) = \int_t^1 \frac{1}{W_2(s)}ds \ \text{ and } \ W_2(s) = sG'(\delta_1 s)$$

for $0 < t \le 1$. Then, by integrating over $[0,t]$, we get,

$$F(t) \le W_1^{-1}\left(k_1\int_0^t \sigma(s)ds + k_2\right), \qquad \forall t \in \mathbb{R}^+ \tag{3.18}$$

The equivalence $F_1 \sim E$ and (3.18) give the desired result.

**Case 2**. $h_0$ is nonlinear on $[0,\varepsilon]$.
First, we assume that $\max\{r_2, h_0(r_2)\} < \varepsilon$; otherwise we take $r_2$ smaller. Let $\varepsilon_1 = \min\{r_2, h_0(r_2)\}$.
Now, using $(A2)$, we have, for $\varepsilon_1 \le |s| \le \varepsilon$,

$$|h(s)| \le \frac{h_0^{-1}(|s|)}{|s|}|s| \le \frac{h_0^{-1}(|\varepsilon|)}{|\varepsilon_1|}|s|$$

and

$$|h(s)| \ge \frac{h_0(|s|)}{|s|}|s| \ge \frac{h_0(|\varepsilon_1|)}{|\varepsilon|}|s|$$

So, we deduce that

$$\begin{cases} h_0(|s|) \le |h(s)| \le h_0^{-1}(|s|) & \text{for all } \ |s| < \varepsilon_1 \\ c_1'|s| \le |h(s)| \le c_2'|s| & \text{for all } \ |s| \ge \varepsilon_1 \end{cases} \tag{3.19}$$

Using (3.19), we get for all $|s| \leq \varepsilon_1$

$$H(h^2(s)) = |h(s)|h_0(|h(s)|) \leq sh(s)$$

which gives

$$h^2(s) \leq H^{-1}(sh(s)) \quad \text{for all} \quad |s| \leq \varepsilon_1 \tag{3.20}$$

To estimate the last integral in (3.8), we define the following partition which was introduced by Komornik [22]:

$$\Omega_1 = \{x \in \Omega : |u_t| > \varepsilon_1\}, \qquad \Omega_2 = \{x \in \Omega : |u_t| \leq \varepsilon_1\}$$

Using (3.19), we get on $\Omega_2$

$$u_t h(u_t) \leq \varepsilon_1 h_0^{-1}(\varepsilon_1) \leq h_0(r_2) r_2 = H(r_2^2) \tag{3.21}$$

Then, with $J(t)$ defined by

$$J(t) := \frac{1}{|\Omega_2|} \int_{\Omega_2} u_t h(u_t) dx$$

Jensen's Inequality gives

$$H^{-1}\left(J(t)\right) \geq c \int_{\Omega_2} H^{-1}(u_t h(u_t)) dx \tag{3.22}$$

Thus, combining (2.9), (3.19) and (3.22), we arrive at

$$\begin{aligned} \sigma(t) \int_{\Omega} h^2(u_t) dx &= \sigma(t) \int_{\Omega_2} h^2(u_t) dx + \sigma(t) \int_{\Omega_1} h^2(u_t) dx \\ &\leq \sigma(t) \int_{\Omega_2} H^{-1}\left(u_t h(u_t)\right) dx + \sigma(t) \int_{\Omega_1} h^2(u_t) dx \\ &\leq c\sigma(t) H^{-1}(J(t)) - cE'(t) \end{aligned} \tag{3.23}$$

Therefore, after multiplying (3.8) by $\sigma(t)$ and using (3.23), we get

$$L_0'(t) \leq -m\sigma(t)E(t) + c\sigma(t)(go\nabla u)(t) + c\sigma(t)H^{-1}(J(t)), \quad \forall t \geq 0 \tag{3.24}$$

where $L_0 = \sigma L + cE$, which is clearly equivalent to $E$.
Now, for $\varepsilon_0 < r_2^2$ and $c_0 > 0$, let

$$L_1(t) := H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) L_0(t) + c_0 E(t)$$

By using the properties of $E$ and $H$, we can conclude that $L_1$ satisifies

$$\begin{aligned} L_1'(t) \leq &-m\sigma(t)E(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\sigma(t)H^{-1}\left(J(t)\right) H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \\ &+ c_0 E'(t) + cH'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)(go\nabla u)(t) \end{aligned} \tag{3.25}$$

Let $H^*$ be the convex conjugate of $H$ in the sense of Young, then

$$\begin{aligned} H^*(s) &= s(H')^{-1}(s) - H[(H')^{-1}(s)] \quad \text{if } s \in (0, H'(r_2)] \\ &\leq s(H')^{-1}(s) \end{aligned} \tag{3.26}$$

Using the general Young inequality:

$$AB \leq H^*(A) + H(B), \quad \text{if } A \in (0, H'(r_2)],\ B \in (0, r_2]$$

for

$$A = H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \qquad \text{and} \qquad B = H^{-1}\left(J(t)\right)$$

we get

$$\begin{aligned} L_1'(t) \leq & -m\sigma(t)E(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0\sigma(t)\frac{E(t)}{E(0)}H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t) \\ & + c_0E(t) + c\sigma(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)(go\nabla u)(t) \\ = & -(mE(0) - c\varepsilon_0)\sigma(t)\frac{E(t)}{E(0)}H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - (c - c_0)E'(t) \\ & + c\sigma(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)(go\nabla u)(t) \end{aligned}$$

Consequently, with a suitable choice of $\varepsilon_0$ and $c_0$. We obtain, for all $t \geq 0$

$$L_2'(t) \leq -k\sigma(t)\frac{E(t)}{E(0)}H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\sigma(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)(go\nabla u)(t) \tag{3.27}$$

where $L_2(t) = L_1(t) + (c - c_0)E(t)$. Multiply (3.27) by $G'\left(\delta_0 E(t)\right)$ and use (3.11), we obtain

(3.28)

$$\begin{aligned} G'\left(\delta_0 E(t)\right) L_2'(t) \leq & -k\sigma(t)\frac{E(t)}{E(0)}G'\left(\delta_0 E(t)\right) H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \\ & - \beta_2\sigma(t)E'(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + \beta_2\delta_0\sigma(t)E(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)G'\left(\delta_0 E(t)\right) \\ \leq & -k\sigma(t)\frac{E(t)}{E(0)}G'\left(\delta_0 E(t)\right) H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - CE'(t) \\ & + \beta_2\delta_0\sigma(t)E(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)G'\left(\delta_0 E(t)\right) \end{aligned}$$

Now, using (3.28) and the fact that $E' \leq 0$ and $G'' > 0$ for $\delta_0 < \frac{r_1}{E(0)}$, we find that the functional $L_3$ defined by

$$L_3(t) := G'\left(\delta_0 E(t)\right) L_2(t) + CE(t)$$

satisfies, for some $\alpha_1, \alpha_2 > 0$

$$\alpha_1 L_3(t) \leq E(t) \leq \alpha_2 L_3(t) \tag{3.29}$$

and

$$L_3'(t) \leq -k\sigma(t)\frac{E(t)}{E(0)}G'\left(\delta_0 E(t)\right)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)$$
$$(3.30) \qquad + \beta_2\delta_0\sigma(t)E(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)G'\left(\delta_0 E(t)\right)$$
$$= -\left(k-\beta_2\delta_1\right)\sigma(t)\frac{E(t)}{E(0)}G'\left(\delta_1\frac{E(t)}{E(0)}\right)$$

where $\delta_1 = E(0)\delta_0$
After choosing $\delta_0$ small enough so that $k_1 := k - \beta_2\delta_1 > 0$, (3.30) becomes
(3.31)

$$L_3'(t) \leq -k_1\sigma(t)\frac{E(t)}{E(0)}G'\left(\delta_1 E(t)\right)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) = -k_1\sigma(t)W_2\left(\frac{E(t)}{E(0)}\right)$$

where $H_2(t) = tH'(\varepsilon_0 t)G'(\delta_1 t)$. Thus, with $R(t) = \frac{\alpha_1 L_3(t)}{E(0)}$ and using (3.29) and (3.31), we have

$$(3.32) \qquad R(t) \sim E(t)$$

and, for some $k_2 > 0$

$$(3.33) \qquad R'(t) \leq -k_2\sigma(t)W_2(R(t))$$

Inequality (3.33) implies that $(W_1(R))' \geq k_2\sigma(t)$, where

$$W_1(t) = \int_t^1 \frac{1}{W_2(s)}ds \quad \text{for } t \in (0,1]$$

Then, by integrating over $[0,t]$, we get

$$(3.34) \qquad R(t) \leq W_1^{-1}\left(k_2\int_0^t \sigma(s)ds + k_3\right), \quad \forall t \in \mathbb{R}^+$$

Finally, we obtain (3.1) by combing (3.32) and (3.34)

**Acknowledgment**
The author thanks KFUPM for its continuous support.

## References

[1] Appleby J., Fabrizio M., Lazzari B. and Reynolds D., *On exponential asymptotic stability in linear viscoelasticity*, Mathematical Models and Methods in Applied Sciences, 16(10), 1677-1694 (2006)
[2] Arnold V., *Mathemtical Methods of Classical Mechanics*, Springer-Verlag, New York, (1989)
[3] Barreto R., Lapa E.C, Muñoz Rivera J.E., *Decay rates for viscoelastic plates with memory*, J. Elasticity 44(1), 61-87 (1996)
[4] Berrimi S., Messaoudi S., *Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping*, Electron. J. Differential Equations. (88) 1-10 (2004)
[5] Berrimi S., Messaoudi S., *Existence and decay of solutions of a viscoelastic equation with a nonlinear source*, Nonl. Anal. 64, 2314-2331 (2006)

[6] Cabanillas E., Muñoz Rivera J., *Decay rates of solutions of an anisotropic inhomogeneous n-dimensional viscoelastic equation with polynomial decaying kernels*, Comm. Math. Phys. 177, 583-602 (1996).

[7] Cavalcanti M., Domingos Cavalcanti V., Ferreira J.,*Existence and uniform decay for nonlinear viscoelastic equation with strong damping*, Math. Methods Appl. Sci. 24, 1043-1053 (2001)

[8] Cavalcanti M., Domingos Cavalcanti V., Martinez P., *General decay rate estimates for viscoelastic dissipative systems*, Nonlinear Anal. 68(1) 177-193 (2008)

[9] Cavalcanti M., Domingos Cavalcanti V., Prates Filho J., Soriano J., *Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping*, Differential Integral Equations 14 (1) 85-116 (2001)

[10] Cavalcanti, M., Domingos Cavalcanti, V., Soriano, J.A, *Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping*, E. J. Differ. Eq. 44, 1-14 (2002)

[11] Cavalcanti M., Oquendo H., *Frictional versus viscoelastic damping in a semilinear wave equation*, SIAM J. Control Optim. 42(4), 1310-1324 (2003)

[12] Conti M., Pata V., *Weakly dissipative semilinear equations of viscoelasticity*, Communications on Pure and Applied Analysis, 4(4), 705-720 (2005)

[13] Dafermos C., *Asymptotic stability in viscoelasticity*, Arch. Ration. Mech. Anal. 37, 297-308 (1970)

[14] Dafermos C., *On abstract volterra equations with applications to linear viscoelasticity*, J. Differ. Equ. 7, 554-569 (1970)

[15] Dasios G., Zafiropoulos F., *Equipartition of energy in linearized 3-D viscoelasticity*, Quart. Appl. Math. 48(4), 715-730 (1990)

[16] Fabrizio M., Morro A., *Mathematical Problems in Linear Viscoelasticity*, SIAM Stud. Appl. Math., vol. 12, Philadelphia, (1992)

[17] Fabrizio M., Polidoro S., *Asymptotic decay for some differential systems with fading memory*, Appl. Anal. 81 (6), 1245-1264 (2002)

[18] Giorgi C., Muñoz Rivera J., Pata V., *Global Attractors for a Semilinear Hyperbolic Equation in viscelasticity*, Journal of Mathematical Analysis and Applications, 260(1), 83-99 (2001)

[19] Guesmia A., *Asymptotic stability of abstract dissipative systems with infinite memory*, J. Math. Appl. 382, 748760 (2011).

[20] Guesmia A., Messaoudi S., *A general decay result for a viscoelastic equation in the presence of past and finite history memories*, Nonlinear Analysis: Real World Applications 13, 476485 477 (2012).

[21] Hrusa, W., *Global existence and asymptotic stability for a semilinear Volterra equation with large initial data*, SIAMJ. Math. Anal. 16(1), 110-134 (1985)

[22] Komornik V., *Exact Controllability and Stabilization. The Multiplier Method*, Masson-John Wiley, Paris (1994)

[23] Lasiecka I., Tataru D., *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping*, Differential Integral Equations 6 (3) 507-533 (1993)

[24] Martinez P., *A new method to obtain decay rate estimates for dissipative systems with localized damping*, Rev. Mat. Complut. 12(1) 251-283 (1999)

[25] Messaoudi S., *General decay of solution energy in a viscoelastic equation with a nonlinear source*, Nonlinear Anal. 69, 2589-2598 (2008)

[26] Messaoudi S., *General decay of solutions of a viscoelastic equation*, J. Math. Anal. App. 341, 1457-1467 (2008)

[27] Messaoudi S., Mustafa M., *On convexity for energy decay rates of a viscoelastic equation with boundary feedback*, Nonlin. Anal. (72), 3602-3611 (2010)

[28] Messaoudi S., Mustafa M., *A general stability result for a quasilinear wave equation*, Nonlin. Anal.(14), 1854-1864 (2013).

[29] Messaoudi S., Tatar N.-E, *Global existence and asymptotic behavior for a non-linear viscoelastic problem*, Math. Methods Sci. Res. 7(4), 136-149 (2003)

[30] Messaoudi S., Tatar N.-E, *Global existence and uniform stability of solutions for a quasilinear viscoelastic problem*, Math. Methods Appl. Sci. 30, 665-680 (2007)

[31] Muñoz Rivera J., *Asymptotic behavior in linear viscoelasticity*, Quart. Appl. Math. 52 # 4, 628-648 (1994).

[32] Muñoz Rivera J., Andrade D., *Exponential decay of nonlinear wave equation with a viscoelastic boundary condition*, Math. Meth. Appl. Sci., 23, 41-61 (2000)

[33] Muñoz Rivera J., Naso N., *Asymptotic stability of semigroups associated to weak dissipative systems with memory*, J. Math. Anal. Appl. 326 # 1, 691-707 (2007)

[34] Muñoz Rivera J., Naso N., Vegni F., *Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory*, J. Math. Anal. Appl. 286 # 2, 692-704 (2003).

[35] Muñoz Rivera J., Oquendo Portillo H., *Exponential stability to a contact problem of partially viscoelastic materials*, J. Elasticity 63 # 2, 87-111 (2001).

[36] Muñoz Rivera J., Peres Salvatierra A., *Asymptotic behavior of the energy in partially viscoelastic materials*, Quart. Appl. Math. 59 # 3, 557-578 (2001).

[37] Pata V., *Stability and exponential stability in linear viscoelasticity*, Milan Journal of Mathematics, 77(1), 333-360, (2009)

[38] Muñoz Rivera, J., Salvatierra, A., *Asymptotic behavior of the energy in partially viscoelastic materials*, Quart. App. Math. 59(3), 557-578 (2001)

[39] Santos M., *Asymptotic behavior of solutions to wave equations with a memory condition at the boundary*, Electron. J. Differential Equations. (73) 1-11 (2001)

Received January 2016; revised October 2017.