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THE POWER QUANTUM CALCULUS AND VARIATIONAL PROBLEMS

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Abstract. We introduce the power difference calculus based on the operator $D_{n,q}f(t) = \frac{f(qt^n)-f(t)}{qt^n-t}$, where n is an odd positive integer and $0 < q < 1$. Properties of the new operator and its inverse — the $d_{n,q}$ integral — are proved. As an application, we consider power quantum Lagrangian systems and corresponding n,q-Euler–Lagrange equations.

Keywords. Quantum variational problems; n,q-power difference operator; generalized Nörlund sum; generalized Jackson integral; n,q-difference equations.

AMS (MOS) subject classification: 39A13; 39A70; 49K05; 49S05.

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