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COMPACTLY SUPPORTED SOLUTIONS OF A FAMILY OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We analyse the existence of solutions with compact support for a family of nonlinear partial differential equations.
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1 Introduction

In this paper we consider equations of the form

$$m_t + ku_x m + um_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

where $m = u - u_{xx}$ and $k \in \mathbb{Z}$. Note that we can write (1) as

$$u_t - u_{txx} + (k+1)uu_x = ku_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{R},\ t \geq 0. \tag{2}$$

If we set $k = 2$ in (2) we get the Camassa-Holm equation for the unidirectional propagation of shallow water waves, with $u(x,t)$ representing the water's free surface over a flat bed in nondimensional variables cf. [2] (see also [17] for an alternative derivation). This equation, first derived as a bi-Hamiltonian equation with infinitely many conservation laws [14], is also a re-expression of geodesic flow on the diffeomorphism group of the line [4, 19]. The Camassa-Holm equation is integrable [2, 5, 9, 14, 20], models wave breaking as well as the propagation of permanent waves [4, 6, 7, 21], and its solitary waves are stable solitons [2, 11]. With $k = 3$ in (2) we get the Degasperis-Procesi equation, an equation which has neither a physical derivation nor a geometrical interpretation, but which is formally integrable [12] and has infinitely many conservation laws.

In this paper, we first prove the local well-posedness of (2) for all $k \in \mathbb{R}$ using Kato's semigroup approach [18]. Then we investigate for which values of $k \in \mathbb{R}$ do the classical solutions m, u of (1) and (2) have compact support for all t, if their initial data has this property. What we show is that for all $k \in \mathbb{R}$ this property holds for the classical solutions m of (1). Furthermore we show that