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SOLITARY WAVES FOR A GENERALIZED CAMASSA-HOLM EQUATION

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Abstract. We establish the existence of solitary wave solutions with exponential decay at infinity for a nonlinear dispersive partial differential equation. As a by-product, the existence of smooth solitary waves for the Camassa-Holm equation, modeling water waves and waves in elastic rods, is obtained.
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1 Introduction

We are concerned here with the nonlinear equation

$$u_t - u_{txx} + [g(u)]_x = \gamma(2u_x u_{xx} + u u_{xxx}), \tag{1}$$

where $g \in C^\infty(\mathbb{R}, \mathbb{R})$, $g(0) = 0$, and $\gamma \in \mathbb{R}$. In this formulation, the equation (1) encompasses several model equations from water wave theory and nonlinear elasticity. If $g(u) = \frac{3}{2}u^2 + 2ku$, $\gamma = 1$, where $k > 0$, the unidirectional propagation of shallow water waves over a flat bottom is described [2]. Here, $u(t,x)$ represents the free surface and k characterizes the critical shallow water speed. The equation was derived either from physical principles [1], [11] or as a bi-Hamiltonian [10]. In this case, the equation (1) is integrable [3]. The formal analysis made in [2] indicates that equation (1) exhibits smooth solitary waves for all $k > 0$ that behave numerically like solitons. The investigations [5], [12] confirm these features of (1). If $g(u) = \frac{3}{2}u^2$, $\gamma \in \mathbb{R}$, the equation (1) models radial deformation waves with small amplitude in thin cylindrical compressible hyperelastic rods [9]. Here, $u(t,x)$ represents the radial stretch relative to a prestressed state. Discussions in the case $g(u) = \frac{3}{2}u^2 + 2ku$, $\gamma \in \mathbb{R}$, can be found in [6], [15]. The Cauchy problem for equation (1) is investigated in [13].

Following [5], a *solitary wave solution* of equation (1), if it exists, should be a C^3- function $u(t,x) = \varphi(x - ct)$, where $c > 0$ is called *propagation speed*, such that $\lim\limits_{s\to\pm\infty} \varphi^{(i)}(s) = 0$ for $i \in \overline{0,3}$. The function φ then verifies