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WEIGHTED SOBOLEV SPACES AND POINCARÉ INEQUALITY FOR DEGENERATE OPERATORS

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Abstract. In this paper we study some properties of the weighted Sobolev space $W^{1,p}_\sigma\left(\mathbb{R}^d,\mu\right)$ proving in particular that its embedding into the space $L^q\left(\mathbb{R}^d,\mu\right)$ is compact for a certain $q<p$. From this result we shall deduce the validity of the Poincaré inequality for some degenerate Kolmogorov operators of gradient type.
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1 Introduction

In this paper we are concerned with the following differential operator

$$N_0\varphi(x) = \operatorname{div}\left(A(x)D\varphi(x)\right) - \left\langle A(x)DU(x), D\varphi(x)\right\rangle,$$

with $\varphi \in C^2_b\left(\mathbb{R}^d\right)$, the space of functions which are continuous and bounded together with their derivatives up to the second order.
The coefficient A is a matrix valued function with values in the space of symmetric and positive *semi*-definite matrices[1], and $U:\mathbb{R}^d \longrightarrow \mathbb{R}$. In the sequel we shall write $A(x)$ in the following more suitable form: $A(x)=\sigma(x)\sigma^*(x)$, and we shall give all our considerations for σ instead that for A.
This type of operators appears quite naturally in mathematical physics, mainly in the non relativistic quantum theory and in the theory of Hamiltonians. It can be proved in fact that the operator $-N_0$ is unitarily equivalent in $L^2\left(\mathbb{R}^d\right)$ to the Schrödinger operator

$$H := -\Delta + V.$$

The main problem is whether such operator, defined on test functions uniquely determines the dynamics, that is if the operator $\left(N_0, C^2_b\left(\mathbb{R}^d\right)\right)$ is symmetric and admits exactly one self-adjoint extension on $L^2\left(\mathbb{R}^d,\mu\right)$, where μ is the probability measure given by

$$d\mu = Z^{-1}e^{-U}\,dx,$$

[1]The determinant of $A(x)$ can be zero for some $x\in\mathbb{R}^d$.