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FREQUENT CONVERGENCE AND APPLICATIONS

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Abstract. The classical concept of limit does not capture the fine details of sequences that do not converge. By means of frequency measures (also called asymptotic densities in the earlier literature) defined for subsets of the set of integers, we define frequent limits and associated concepts for real sequences. These concepts are more general than the classical ones and their properties we obtained enable us to better handle the complex asymptotic behaviors of dynamical systems.
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1 Introduction

A real sequence $\{x_k\}_{k=1}^\infty$ is said to converge to L if given any $\varepsilon > 0$, there is a real number $N = N_\varepsilon$ such that $|x_n - L| < \varepsilon$ for all $n \geq N$. However, the above definition does not capture the fine details of sequences that do not converge to L. For this reason, superior and inferior limits are introduced. But these definitions are not the only remedies. This can be seen from considering the following two sequences $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ defined by

$$\alpha_n = \begin{cases} -1/n & n \neq 10, 10^2, 10^3, \cdots \\ 1 & \text{otherwise} \end{cases}, \tag{1}$$

and

$$\beta_n = \begin{cases} -1/n & n \neq 1, 3, 5, \cdots \\ 1 & \text{otherwise} \end{cases}. \tag{2}$$

Indeed, although they have the same superior and inferior limits, it appears that the sequence $\{\beta_n\}$ is near to 1 more 'frequently' than the sequence $\{\alpha_n\}$. It is possible for us to use the well known asymptotic density in number theory to describe this phenomenon. From the literature, a sequence $A = \{a_n\}_{n=1}^\infty$ of positive integers $a_1 < a_2 < \cdots$ has lower asymptotic density $\underline{\delta}(A)$ and upper asymptotic density $\overline{\delta}(A)$ defined by

$$\underline{\delta}(A) = \liminf_{n\to\infty} \frac{A(n)}{n}, \quad \overline{\delta}(A) = \limsup_{n\to\infty} \frac{A(n)}{n},$$