THE INTERACTION OF PEAKONS AND ANTIPEAKONS

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Abstract. In this note we discuss some aspects of the peakon-antipeakon interaction for the Camassa-Holm equation.

Keywords. Shallow water equation, Weak solutions, Solitons, Hamiltonian system, Wave breaking.

AMS (MOS) subject classification: 35L05, 35Q35.

1 Introduction

The equation

\[ \begin{cases} u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & t > t_0, \ x \in \mathbb{R}, \\ u(t_0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \]  

(1.1)

was discovered by Fokas and Fuchssteiner [16] as a bi-Hamiltonian system with infinitely many conservation laws. It was later obtained as a model for shallow water waves, \( u(t, x) \) representing in nondimensional variables the water’s free surface above a flat bottom, by Camassa and Holm [2] (see also [17] for a discussion of the physical relevance of (1.1) in the context of shallow water waves). Among the other interesting aspects of (1.1), we would like to point out that it is completely integrable [4, 10] and that it is a reexpression of geodesic flow on the diffeomorphism group of the line (see [8, 9, 20]).

Equation (1.1) has solitary waves of the form

\[ u_c(t, x) = c\varphi(x - ct), \quad x \in \mathbb{R}, \]

where \( \varphi(x) = e^{-|x|} \). We call \( u_c \) a peakon if \( c > 0 \) and an antipeakon if \( c < 0 \). These waves have peaked corners at \( x = ct \), and therefore have to be interpreted as weak solutions, using the fact that (1.1) can be reformulated as the nonlocal conservation law

\[ u_t + uu_x + \partial_x p * \left( u^2 + \frac{1}{2}u_x^2 \right) = 0, \]  

(1.2)

where \( p(x) := \frac{1}{2}e^{-|x|} \).

The solitary waves are solitons, meaning that they retain their individuality under interaction and eventually emerge with their original shapes and