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SPINODAL DECOMPOSITION FOR SPATIALLY DISCRETE CAHN-HILLIARD EQUATIONS†

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Abstract. The process of phase separation with a characteristic wavelength is a phenomenon known as spinodal decomposition. In this paper, an extensive and mathematically rigorous analysis is performed for how and when spinodal decomposition occurs for a spatially discretized fourth-order parabolic partial differential equation known as the Cahn-Hilliard equation. First, a linearization is considered for the spatially discrete equation. It is shown that the unstable eigenvalues for the discrete linear equation are almost equal to the eigenvalues of the continuous linear equation for a sufficiently fine discretization of the domain. Then we show that, with a probability close to one, an initial condition, chosen at random inside a particular neighbourhood of an homogeneous equilibrium in the spinodal interval, will lead to spinodal decomposition for the discrete problem. An estimate of the wavelength of spinodally decomposed states is also derived.

1. Introduction. When a high-temperature homogeneous mixture of two metals is quenched to a lower temperature, the mixture may exhibit a phase separation which will occur in two stages. In the first stage, the mixture quickly becomes inhomogeneous as it decomposes into a fine-grained structure, which exhibits a characteristic length scale. This phenomenon is known as spinodal decomposition. Following this stage, the mixture will go through a coarsening process in which the characteristic length scale grows. Cahn and Hilliard [6, 10] proposed a fourth-order parabolic partial differential equation, which describes this process of phase separation and is given by

$$u_t = -\Delta(\epsilon^2 \Delta u + f(u)), \quad \text{for all } x \in \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0, \quad \text{for all } x \in \partial\Omega,$$

where ν is the unit outward normal, ϵ is a small parameter, and $-f$ is the derivative of a double-well potential W, the standard example being the nonlinear cubic function $f(u) = u - u^3$. Here and throughout the paper, Δ denotes the standard Laplacian, and $u_t = \partial u/\partial t$. In general $\Omega \subset \mathbb{R}^d$ is a

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