NONLINEAR ERGODIC THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

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Abstract. In this paper, we study nonlinear ergodic properties for an asymptotically nonexpansive semigroup in a Banach space. We prove that if \( S \) is amenable and \( S = \{ T_t : t \in S \} \) is an asymptotically nonexpansive semigroup on a nonempty closed convex subset \( C \) of a uniformly convex Banach space \( E \) such that the set \( F(S) \) of common fixed points of \( S \) is nonempty, then there exists a nonexpansive retraction \( P \) of \( C \) onto \( F(S) \) such that \( PT_t = T_t P = P \) for every \( t \in S \) and \( Px \in \text{co}\{T_t x : t \in S\} \) for every \( x \in C \). Also, if the norm of \( E \) is Fréchet differentiable, then for each \( x \in C \), \( Px \) is the unique common fixed point in \( \bigcap_{t \in S} \text{co}\{T_t x : t \in S\} \). Further, if \( \{ \mu_\alpha \} \) is an asymptotically invariant net of means, then for each \( x \in C \), \( \{T_{\mu_\alpha} x\} \) converges weakly to \( Px \). Finally, we provide a necessary and sufficient condition for the existence of such a retraction \( P \).

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1. Introduction

Let \( C \) be a nonempty closed convex subset of a real Banach space \( E \). Then, a mapping \( T : C \to C \) is said to be Lipschitzian if there exists a nonnegative real number \( k \) such that

\[
\|Tx - Ty\| \leq k\|x - y\| \quad \text{for every } x, y \in C.
\]

\( T \) is said to be nonexpansive if \( k = 1 \). Let \( S \) be a semigroup. Then, a family \( S = \{ T_t : t \in S \} \) of mappings from \( C \) into itself is said to be a Lipschitzian semigroup on \( C \) with Lipschitz constants \( \{ k_t : t \in S \} \) if it satisfies the following:

\( 1 \). For each \( t \in S \), there exists a nonnegative real number \( k_t \) such that

\[
\|T_t x - T_t y\| \leq k_t \|x - y\| \quad \text{for every } x, y \in C;
\]

\( 2 \). \( T_s T_t x = T_{st} x \) for every \( s, t \in S \) and \( x \in C \).

We denote by \( F(S) \) the set \( \{x \in C : T_t x = x \text{ for every } t \in S\} \) of common fixed points of \( S \). We know that if \( E \) is uniformly convex and \( \inf_t \sup_s k_{ts} \leq 1 \), then \( F(S) \) is closed and convex; see [23] for details. \( S \) is said to be a nonexpansive semigroup on \( C \) if \( k_t = 1 \) for every \( t \in S \). \( S \) is also said to be an asymptotically nonexpansive semigroup on \( C \) if \( \inf_t \sup_s k_{ts} \leq 1 \) and \( \sup_t k_t < \infty \). In particular, \( S \) is said to be a one-parameter asymptotically