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NONLINEAR ERGODIC THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

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Abstract. In this paper, we study nonlinear ergodic properties for an asymptotically nonexpansive semigroup in a Banach space. We prove that if S is amenable and $\mathcal{S} = \{T_t : t \in S\}$ is an asymptotically nonexpansive semigroup on a nonempty closed convex subset C of a uniformly convex Banach space E such that the set $F(\mathcal{S})$ of common fixed points of $\mathcal{S}$ is nonempty, then there exists a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{\text{co}}\{T_tx : t \in S\}$ for every $x \in C$. Also, if the norm of E is Fréchet differentiable, then for each $x \in C$, Px is the unique common fixed point in $\bigcap_{s\in S} \overline{\text{co}}\{T_{ts}x : t \in S\}$. Further, if $\{\mu_\alpha\}$ is an asymptotically invariant net of means, then for each $x \in C$, $\{T_{\mu_\alpha}x\}$ converges weakly to Px. Finally, we provide a necessary and sufficient condition for the existence of such a retraction P.

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1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E. Then, a mapping $T : C \to C$ is said to be Lipschitzian if there exists a nonnegative real number k such that

$$\|Tx - Ty\| \leq k\|x - y\| \text{ for every } x, y \in C.$$

T is said to be nonexpansive if $k = 1$. Let S be a semigroup. Then, a family $\mathcal{S} = \{T_t : t \in S\}$ of mappings from C into itself is said to be a Lipschitzian semigroup on C with Lipschitz constants $\{k_t : t \in S\}$ if it satisfies the following:

(1) For each $t \in S$, there exists a nonnegative real number k_t such that

$$\|T_tx - T_ty\| \leq k_t\|x - y\| \text{ for every } x, y \in C;$$

(2) $T_{st}x = T_sT_tx$ for every $s, t \in S$ and $x \in C$.

We denote by $F(\mathcal{S})$ the set $\{x \in C : T_tx = x \text{ for every } t \in S\}$ of common fixed points of $\mathcal{S}$. We know that if E is uniformly convex and $\inf_s \sup_t k_{ts} \leq 1$, then $F(\mathcal{S})$ is closed and convex; see [23] for details. $\mathcal{S}$ is said to be a nonexpansive semigroup on C if $k_t = 1$ for every $t \in S$. $\mathcal{S}$ is also said to be an asymptotically nonexpansive semigroup on C if $\inf_s \sup_t k_{ts} \leq 1$ and $\sup_t k_t < \infty$. In particular, $\mathcal{S}$ is said to be a one-parameter asymptotically