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ON A CLASS OF NONLINEAR ELLIPTIC BVP AT CRITICAL GROWTH

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Abstract. Existence and multiplicity of solutions for a general class of quasilinear elliptic problems at critical growth with perturbations of lower order are considered. Techniques of nonsmooth critical point theory are employed.
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1 Introduction and main results

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $1 < p < n$. This paper is devoted to the existence and multiplicity of solutions in $W_0^{1,p}(\Omega)$ to the following class of nonlinear elliptic BVP, say $\mathscr{C}_g$, involving the critical Sobolev exponent $p^* = \frac{np}{n-p}$

$$\begin{cases} -\operatorname{div}\,(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = |u|^{p^*-2}u + g(x,u) & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega. \end{cases}$$

As a particular case, for $p < q < p^*$, we consider problems, say $\mathscr{C}_{\varepsilon,\lambda}$,

$$\begin{cases} -\operatorname{div}\,(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = |u|^{p^*-2}u + \lambda|u|^{q-2}u + \varepsilon h & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega \end{cases}$$

with $h \in L^{p'}(\Omega)$, $h \neq 0$, $\lambda > 0$ and $\varepsilon > 0$.

Motivations for investigating $\mathscr{C}_g$ and $\mathscr{C}_{\varepsilon,\lambda}$ come from various situations in geometry and physics which involve lack of compactness (see e.g. [5]). Since, as known, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ fails to be compact, one encounters serious difficulties in applying variational methods.

We refer the reader to [5] for the case $j = -\Delta$ and to [3, 9, 10] for the extension to degenerate operators ($j = -\Delta_p$, $p \neq 2$). For the existence of multiple solutions (two) for problems $\mathscr{C}_{\varepsilon,\lambda}$, we refer the reader to [17] for $j = -\Delta$ and to [8] for $j = -\Delta_p$. In these cases the associated functional is smooth (C^1).

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