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MIXED CONVEXITY - CONCAVITY PROPERTIES OF SOLUTIONS OF DIFFERENTIAL EQUATIONS RELATIVE TO INITIAL VALUES

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Abstract. In this paper, we show that mixed convexity-concavity properties of f result in similar convexity-concavity properties of the solutions.

1 Introduction

Interesting results on the convex dependence of solutions of differential equations relative to the initial data were recently obtained. First, Sarychev [5] showed that the solutions $x(t;t_0,x_0)$ of the IVP, $x' = f(t,x)$, $x(t_0) = x_0$, are convex relative to x_0 if f is convex in x, uniformly in t. Then, a simplified proof of this result, as well as its extension to IVP's in a Banach space, were given by Lakshmikantham et al. [3, 6] and in a very recent paper [2], it was shown that under the same conditions on f, convexity of the solution relative to t_0 also holds, thereby establishing the fact that convexity of f relative to x implies the convexity of the solutions relative to the initial data (t_0, x_0). In this paper, we show that mixed convexity-concavity properties of f result in similar convexity-concavity properties of the solutions.

2 Preliminaries

Let $f\colon L \to \mathbb{R}$ be a real-valued function defined on an arbitrary normed linear space L, and let $U = B(0,b)$ be a convex subset of L. Then, for $x_1, x_2 \in U$, $\delta \in (0,1)$, $sx_1 + (1-s)x_2 \in U$ and $\| x \| < b$ for any $x \in U$. First we state the following definitions and theorems on convexity for future reference. For more details, see [4].

Definition 2.1: $f\colon L \to \mathbb{R}$ is convex on $U \subseteq L$ if

$$f(sx_1 + (1-s)x_2) \leq sf(x_1) + (1-s)f(x_2) \tag{2.1}$$

where $x_1, x_2 \in U$, $s \in (0,1)$. If $(-f)\colon L \to \mathbb{R}$ is convex on U, then f is said to be concave on U.

On an infinite-dimensional space, it is generally not the case that a convex function is continuous. However, a convex function defined on an open convex subset U of $\mathbb{R}^n$ is continuous. This is stated in the following theorem.