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CHARACTERIZING AND COMPUTING NORMAL FORMS USING LIE TRANSFORMS: A SURVEY

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Abstract. The normal form is an important tool in the study of dynamical and bifurcations problems. In practice, the computation of normal forms requires a great amount of operations, so that the use of adequate tools becomes essential.

In this paper, we present the main results (both, computational and theoretical) obtained by approaching this problem from the Lie transforms perspective.

The Poincaré-Dulac-Birkhoff normal form theorem determines how much a vector field can be simplified, depending uniquely on its linear part. Nevertheless, taking into account the nonlinear terms, it is possible to obtain further simplifications in the classical normal form. Our approach is not only useful to get algorithms for computing these simplified normal forms, but also to characterize them.

The cases of Hopf, Takens-Bogdanov, Hopf-zero and triple-zero singularities, are presented, including the improvements achieved by using $\mathcal{C}^\infty$-equivalence.

Keywords. Simplest normal forms, Lie transforms.

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1 Introduction

The normal form theory is a powerful tool for the analysis of dynamical and bifurcation problems near a nonhyperbolic equilibrium point. The idea underlying in this theory is to use near-identity transformations to remove, in the analytic expression of the vector field, the terms that are unessential in the local dynamical behaviour.

The normal form theorem determines classes of vector fields as simple as possible, depending upon the linear part (which characterizes the homological operator). The key of the problem of simplifying the terms of a given order k is to solve the homological equation (see (6) later). This linear equation has, in general, a nonunique solution, that will depend on an arbitrary additive term belonging to the kernel of the homological operator. Hence, a number of arbitrary constants will appear in the expression of the solution and, consequently, in the normal form of order greater than k. These constants can be used, depending on the form of the nonlinear terms, to obtain further simplifications in the normal form, leading to the concept of *unique normal forms* or *hypernormal forms.*